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## Summary

We consider identification of operator families defined via a time-frequency series expansion of the operator spreading function. The identification problem is transformed into an infinite-dimensional linear algebra problem. Our aim is to establish a connection between the identifiability of the operator family and a density measure of the time-frequency index set. In this way, the identification problem can be compared to the classical density condition for existence of Gabor frames. The conclusion is that the relationship between identifiability of such operator families and the "critical" density is highly intricate because of the presence of additional conditions. Criteria for identifiability are developed for families of time-frequency localized operators defined via time-frequency series expansions of the spreading function based on the Gaussian function.

# Contents

# 1 Introduction

The main purpose of this study is to explore identification of operators having a time-frequency representation. The goal of operator identification is to recover an incompletely known operator from a given operator family through observation of a single input and output result. In general for normed linear spaces $X, Y$ and $\mathcal{H} \subset \mathcal{L}(X, Y)$, we wish to find an element $f \in X$ such that the evaluation map $\Phi_f : \mathcal{H} \to Y$ is bounded and stable. Then $\mathcal{H}$ is said to be identifiable by $f$. Such problems have been considered in mobile radio communications.

Hilbert-Schmidt operators can be represented as a superposition of time-frequency shift operators $T_t M_\nu$:

$$Hf(x) = \iint \eta_H(t,\nu) T_t M_\nu f(x) d(t,\nu),$$

where $\eta_H$ is the spreading function of the operator. The spreading function defines the operator uniquely, and each Hilbert-Schmidt operator has a unique spreading function [Grö01, KP06].

We study classes of operators defined via time-frequency (Gabor) representations of the operators' spreading functions. Namely, our point of interest are operator families of the following type

$$\mathcal{H}_\Lambda = \{H : \eta_H \in \overline{\operatorname{span}}\, \{\pi(\lambda)\eta_0, \lambda \in \Lambda\}\},$$

with $\Lambda$ a lattice in $\mathbb{R}^{4d}$, $\eta_0$ a window function in some subspace of $L^2(\mathbb{R}^{2d})$. We denote the operator having a spreading function $\pi(\lambda)\eta_0$ by $H_\lambda$. The Gabor expansion in $L^2(\mathbb{R}^d)$,

$$\eta_H = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)\eta_0$$

can be translated onto the Hilbert-Schmidt space into the following series representation of a member of $\mathcal{H}$

$$H = \sum_{\lambda\in\Lambda} c_\lambda H_\lambda.$$

Two important criteria for identification of $\mathcal{H}$ are therefore: the structure of index set $\Lambda$ (parametrization in $\mathbb{R}^{4d}$); the properties of $\eta_0$ and its associated prototype operator $H_0$.

Our approach to identification aims at recovering the coefficients $\{c_\lambda\}$ of the Gabor series expansion of $\eta_H$. This is achieved by a discretization of the above in terms of a linear system dependent on $\eta_0$ and $\Lambda$. That is why it is important that the coefficient $\{c_\lambda\}$ must uniquely correspond to $\eta_H$. In other words, we require that $(\eta_0, \Lambda)$ is a Riesz sequence inside $L^2(\mathbb{R}^{2d})$. The meaningfulness of this condition is illustrated in Section 4.1. It allows us to relate the Hilbert-Schmidt norm of an operator $H \in \mathcal{H}_\Lambda$ to the $\ell^2$-norm of the coefficients of the expansion of $\eta_H$ in terms of $\{\pi(\lambda)\eta_0 : \lambda \in \Lambda\}$, or to the coefficients of the expansion of $H$ in terms of $H_\lambda$. That is,

$$\|H\|_{HS} = \left\|\sum_\lambda c_\lambda H_\lambda\right\|_{HS} = \left\|\sum_\lambda c_\lambda \pi(\lambda)\eta_0\right\|_2 \asymp \|\mathbf{c}\|_{\ell^2}.$$

In order to show that the operator family $\mathcal{H}$ is identifiable, we must find $f$ such that the family $\{H_\lambda f : \lambda \in \Lambda\}$ is a Riesz sequence in $L^2(\mathbb{R}^d)$. Such a requirement will imply the validity of the norm equivalence

$$\|Hf\|_2 = \left\|\sum_\lambda c_\lambda H_\lambda f\right\|_2 \asymp \|\mathbf{c}\|_{\ell^2}$$

Then the norm equivalence between the Hilbert-Schmidt norm of $H$ and the $L^2$-norm of $Hf$ will prove identifiability with $f$.

For proving non-identifiability it is enough to show that the mapping $\Phi_f : H \to Hf$ is non-invertible for any $f$ in a particular modulation space (the dual space of the Feichtinger algebra $S_0$, $M^\infty(\mathbb{R}^d)$, which contains the Dirac delta.)

We must stress that the operators act on distributions from $d$-dimensional space, while the spreading functions of the operators are from a $2d$-dimensional space. Therefore, a single evaluation $\Phi_f(H)$ can not determine a general operator $H$ (problem of 'dimension-counting'), and we have to assume some **a priori** knowledge of $\mathcal{H}$. We shall assume that the index set $\Lambda$ of the Gabor system $(\eta_0, \Lambda)$ is $2d$-dimensional. Our main goal is to relate the identifiability of the respective operator class to some measure of density of the index set $\Lambda$. In [KP06] identifiability of a particular operator class is related to the measure (area) of the support of the spreading function $\eta_H$. Underspread operators are those where the area of the support of $\eta_H$ is less than 1, and overspread else. This dichotomy is modeled after the under- and oversampling in Gabor analysis.

By a nalogy we shall define a 'Beurling-type' 2-density for sets of points $\Lambda$ lying within general $2d$-subspaces $\mathbb{S}$ of $\mathbb{R}^{4d}$ - see Definition 4.2. We shall restrict our attention to lattices which define a $2d$-dimensional hyperplace of $\mathbb{R}^{4d}$. For $d = 1$ they are defined by a strictly rectangular $4 \times 2$-matrix,

$$\Lambda = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} \mathbb{Z}^2 = \begin{bmatrix} a_1 m + a_2 n \\ b_1 m + b_2 n \\ c_1 m + c_2 n \\ d_1 m + d_2 n \end{bmatrix} : m, n \in \mathbb{Z}.$$

Since such $\Lambda$ are parametrized by 2 indices, the identification problem becomes well-posed. The *2-density* of $\Lambda$ is then

$$\begin{aligned} D_{(2)}(\Lambda) = & [(a_1b_2 - a_2b_1)^2 + (a_1c_2 - a_2c_1)^2 + (a_1d_2 - a_2d_1)^2 + (b_1c_2 - b_2c_1)^2 \\ & + (b_1d_2 - b_2d_1)^2 + (c_1d_2 - c_2d_1)^2]^{-1/2}. \end{aligned}$$

Some standard choices of matrix coefficients $a_i, b_i, c_i, d_i, i = 1, 2$ are listed in Figures 2 and 3. For higher dimensions densities can be defined analogously.

A reasonable assumption for the criterion for identifiability of $\mathcal{H}_\Lambda$ is the magnitude of the 2-density of $\Lambda$ (this criterion is given also in [KP06] in other terms - *critical spread of the operator*). An underlying idea is that whenever the $D_2(\Lambda)$ is high, the information about the operator is densely packed in the coefficients of its time-frequency representation, and identification is not possible. Our working hypothesis is thus formulated as follows:

*There exist constants $c, C > 0$ such that*

$$D_2(\Lambda) > C \implies \mathcal{H}_\Lambda \text{ is not identifiable.} \tag{1.1}$$

$$D_2(\Lambda) < c \implies \mathcal{H}_\Lambda \text{ is identifiable.} \tag{1.2}$$

Our approach to determining identifiability by means of discretization is the following. To prove that $\mathcal{H}_\Lambda$ is identifiable, we search for an identifier (a distribution $f$) such that any choice of coefficients $c_\lambda$ from (some subclass of) $\ell^2(\mathbb{Z}^4)$, can be computed from $Hf$. Equivalently $c_\lambda$ can be computed from the values of the inner products $v_\mu = \langle Hf, \pi(\mu)\gamma\rangle, \mu \in \mathcal{M}$, which are the Gabor coefficients of $Hf$ with respect to a Gabor frame $(\gamma, \mathcal{M})$ for $L^2$. Hence, we need to solve the system of equations

$$v_\mu = \langle Hf, \pi(\mu)\gamma\rangle = \sum_{\lambda\in\Lambda} c_\lambda \langle H_\lambda f, \pi(\mu)\gamma\rangle = \sum_{\lambda\in\Lambda} c_\lambda A_{\mu;\lambda}. \tag{1.3}$$

The linear system (1.3) has a matrix-vector representation $Ac = v$, where

$$A = (A_{\mu;\lambda})_{\mu;\lambda}; \quad A_{\mu;\lambda} = \langle H_\lambda f, \pi(\mu)\gamma\rangle.$$

If there exists $f$ such that the map $A\colon Y \to \ell^2(\mathbb{Z}^2), Y \subset \ell^2(\mathbb{Z}^4)$ is invertible, then $\mathcal{H}_\Lambda$ is identifiable. On the other hand, if for every $f$ belonging to a particular space of distributions (for example, the modulation space $M^\infty$), the map $A$ is not invertible, then $\mathcal{H}_\Lambda$ is not identifiable with identifiers from this space.

Numerical evidence is given in Section 4, where we explores the existence of bounds $c, C$ as described above for different $\Lambda$. Numerical examples show that the factors for identification (2-density, shape of $\Lambda$) have different importance in determining whether $\mathcal{H}_\Lambda$ is identifiable or not. Our results show that an lower bound on 2-density of $\Lambda$ is not always necessary. Membership of $\eta_0$ in certain modulation shapes is sometimes strong enough to show non-identifiability. For instance, when $\eta_0 \in M^1_v(\mathbb{R}^2)$, there exists lattices $\Lambda$ such that the operator family $\mathcal{H}_\Lambda$ is never identifiable (Propositions 4.8 and 4.9).

In other cases bounds on 2-density of $\Lambda$ are nonetheless an important factor. If $\eta_0$ belongs to a certain modulation space, for different $\Lambda$, the constants $c, C$ do play a role. Such examples are provided by Proposition 4.10 and 4.11. However, in each of these cases the lower bound $C$ is different because the respective index sets $\Lambda$ are different.

To explore the behavior of upper bound $c$, we keep $\eta_0$ fixed and vary $\Lambda$. Again for some $\Lambda$, there exists $c$ such that $D_2(\Lambda) < c$ implies identifiability of $\mathcal{H}$ (Proposition 4.14, 4.20 and Corollary 4.16). On the other hand, we note that a *universal* upper bound $c$ does not exist; in some cases, as Proposition 4.12 illustrates, even families with 2-density close to 0 are not identifiable. In fact, sometimes the set of values of $D_2(\Lambda)$ for which the respective $\mathcal{H}_\Lambda$ is identifiable is not even connected in $\mathbb{R}$, namely for all $c > 0$, the interval $[0, c)$ contains infinitely many values such that $\mathcal{H}_\Lambda$ is *not* identifiable.

Furthermore, we provide examples (Proposition 4.15, 4.21, 4.22, 4.23, 4.24) demonstrating that extra conditions on the parameters of $\Lambda$ besides 2-density are required in order for the identification problem to even make sense. These cannot be formulated in terms of a single numerical criterion such as 2-density.

Since the interplay between all these factors is so difficult to grasp, Section 5 considers the identification problem from a different angle for $d = 1$. It explores the admissible range of upper bounds $C$ for $D_2(\Lambda)$ such that the identification problem is well-posed and makes sense. An example is provided of $\Lambda$ with $D_2(\Lambda) > 1$ and $\mathcal{H}_\Lambda$ identifiable, which shows that $C \geq 1$.

Furthermore, we demonstrate that in the most general case in order for the operator families parametrized by general lattices in $\mathbb{R}^4$ to be identifiable, $C$ must not exceed $\sqrt{2}$ (Theorem 5.8). That is why we pose as universal bound for non-identifiability of $\mathcal{H}_\Lambda$, $\Lambda$ - 2-dimensional index set, $C = \sqrt{2}$ (Theorem 5.9). Our method of proof involves the theory of Gabor molecules [BCHL06a], [BCHL06b]. This is to our knowledge first application of Gabor molecules beyond the problem of measuring localization properties of the elements of the dual frame, which is the main interest for the authors of [BCHL06a], [BCHL06b].

The examples considered show that the identification problem can not be formulated in a straightforward way similar to that of a density condition in the problem of existence of Gabor frames. Identification of operator families even with severe restrictions on the spreading functions involves a lot more than simple density estimates - as evident in the case of higher-dimensional Gabor systems with Gaussian windows.

# 2 Preliminaries

In this section we make an overview of the theoretical background used in the paper. First we do a brief overview of some general properties of Riesz bases and frames for a separable Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ because these are essential ingredients in our analysis. Then we define modulation spaces and present their most important properties, as well as applications to Gabor analysis and define Hilbert-Schmidt operators for $L^2$ as well as their extension to distribution spaces.

## 2.1 Gabor frames and Riesz bases

In the following paragraphs we recall the most important concepts from Riesz basis and frame theory. A sequence $\{f_j\} \subset \mathcal{H}$ is a *Riesz sequence* if and only if there exist constants $a, b > 0$ such that for all finitely supported sequences of scalars $\{c_j\}_{j\in\mathbb{N}}$,

$$a \sum_{j\in\mathbb{N}} |c_j|^2 \leq \Big\| \sum_{j\in\mathbb{N}} c_j f_j \Big\|^2 \leq b \sum_{j\in\mathbb{N}} |c_j|^2 \tag{2.1}$$

A *Riesz basis* for $\mathcal{H}$ is a Riesz sequence whose linear span is complete in $\mathcal{H}$. (2.1) shows that Riesz sequences are more general type of bases than ONB.

A sequence $\mathcal{F} = \{f_j\}_{j\in\mathbb{N}}$ is a *frame* for $\mathcal{H}$ if there exist $0 < a \leq b$ such that for all $f \in \mathcal{H}$,

$$a\|f\|^2 \leq \sum_{j\in\mathbb{N}} |\langle f, f_j\rangle|^2 \leq b\|f\|^2 \,. \tag{2.2}$$

A *frame sequence* is a frame for the closure of its linear span.

The constants $0 < a \leq b$ are called *lower* and *upper frame bound* respectively. A frame is called *tight* if we can choose $a = b$. If $a = b = 1$, the frame is called a *Parseval tight frame.*

**Definition 2.1** *Let $\Lambda \subset \mathbb{R}^{2d}$ be a discrete set. A Gabor system $(g, \Lambda)$ for $L^2(\mathbb{R}^d)$ is the set of all time-frequency shifts of the window function $g$ by $\lambda = (x, \omega) \in \Lambda$, i.e.*

$$(g, \Lambda) := \{g_\lambda : \lambda \in \Lambda\},$$

*for $g_\lambda(t) = \pi(\lambda)g(t) = T_x M_\omega g = g(t - x)e^{2\pi i\langle\omega,t\rangle}$*

We define the short-time Fourier transform with window $g$ as

$$V_g f(t, \nu) = \int_{\mathbb{R}^d} f(x)\overline{g(x - t)}e^{-2\pi i\langle\nu,x-t\rangle}\, dx.$$

The map $V_g$ is central in Gabor analysis - for a discussion of its properties we refer to [Grö01].

We outline the basic definitions:

- A Gabor system $(g, \Lambda)$ is a *Riesz basis sequence* if there exist constants $0 < a \leq b$ such that for all $\boldsymbol{c} \in \ell^2(\Lambda)$,

$$a\|\boldsymbol{c}\|_{\ell^2}^2 \leq \| \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g\|_2^2 \leq b\|\boldsymbol{c}\|_{\ell^2}^2 \,. \tag{2.3}$$

- A *Gabor Riesz basis* is a Riesz basis for $L^2(\mathbb{R}^d)$ if it is also complete in $L^2(\mathbb{R}^d)$.

- A Gabor system $(g, \Lambda)$ is a *frame* for $L^2(\mathbb{R}^d)$ with frame bounds $0 < a \le b$ if such that for all $f \in L^2(\mathbb{R}^d)$,

$$a\|f\|^2 \le \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \le b\|f\|^2\,. \tag{2.4}$$

- A *Gabor frame sequence* is a frame for the $L^2$-closure of its linear span.

The operator

$$S_{(g,\Lambda)} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d); \quad S_{(g,\Lambda)} : f \mapsto \sum V_g f(\lambda)\, \pi(\lambda) g$$

is called a *Gabor frame operator.* It is a positive, bounded, invertible and self-adjoint operator if $(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$.

When $\Lambda$ is a regular lattice, the frame operator $S_{g,\Lambda}$ commutes with the time-frequency shifts $\{\pi(\lambda), \lambda \in \Lambda\}$ [Chr03]. This property of the frame operator underlies the fundamental observation that the dual frame of a Gabor frame on a regular lattice has the structure of a Gabor frame with the same lattice. Gabor frames posses therefore a very useful reconstruction formula:

$$f = \sum_{\lambda\in\Lambda} V_g f(\lambda)\, \pi(\lambda)\gamma = \sum_{\lambda\in\Lambda} V_\gamma f(\lambda)\, \pi(\lambda) g, \tag{2.5}$$

where $\gamma$ is the (canonical) dual window. For a detailed discussion of further properties of Gabor frames, their duals and the Gabor frame operator we refer to [FK98], [FZ98], [Chr03], [Grö01].

The following theorem shows that symplectic transformations of the lattice leave the Gabor frame property ‘invariant’. It is an important tool for verifying the frame or Riesz sequence property of a Gabor system from a known Gabor frame or Riesz sequence.

**Theorem 2.2** *Let $\Lambda$ be a full rank lattice in $\mathbb{R}^{2d}$ and $M \in \mathrm{Sp}(d)$. Then the following are equivalent:*

1. *There exists a $g \in L^2(\mathbb{R}^d)$ such that $(g, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ (respectively Riesz sequence).*

2. *There exists a $\tilde{g} \in L^2(\mathbb{R}^d)$ such that $(\tilde{g}, M\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ (respectively Riesz sequence).*

*Remark:* The window $\tilde{g} = \mu(M)g$, where $\mu(M)$ is the metaplectic operator associated to $M$.

In our analysis of the identification problem we shall make frequent use of functions and distributions belonging to certain modulation spaces. We recall briefly in Section 2.2 their basic properties.

## 2.2 Modulation spaces

Modulation spaces are useful tools in time-frequency analysis because they allow characterization of time-frequency properties of functions via membership in certain Banach spaces. In particular we employ prototype spreading functions $\eta_0$ from weighted modulation spaces in order to define operators $H_0$ with time-frequency localization property (Lemma 4.4).

In this section we introduce the basic properties of modulation spaces, starting from the definition of weight functions and Wiener amalgam spaces. These describe the decay and growth of functions and will be applied to the definitions of modulation spaces.

**Definition 2.3** *A weight function $m$ is a non-negative, locally integrable function on $\mathbb{R}^d$. Two weight functions $m_1, m_2$ are called equivalent if there exists a constant $C > 0$ such that $\frac{1}{C}m_1(\boldsymbol{z}) \leq m_2(\boldsymbol{z}) \leq Cm_1(\boldsymbol{z})$ for all $\boldsymbol{z} \in \mathbb{R}^d$.*

The standard weight functions used in Gabor analysis are polynomial. They will be denoted by $v_s(z) := (1 + |z|)^s$. If $z = (x, \omega) \in \mathbb{R}^{2d}$, then $v_s$ is equivalent to the weights $v'_s = (1 + |x| + |\omega|)^s$ and $v''_s = (1 + x^2 + \omega^2)^{s/2}$ [Grö01].

Wiener amalgam spaces allow 'a separation of local and global properties of a function or distribution' [FZ98]. Here $A(\mathbb{R}^d) = \mathcal{F}L^1(\mathbb{R}^d)$.

**Definition 2.4** *Let $\psi \in A(\mathbb{R}^d)$ be compactly supported and generate a partition of unity, that is*

$$\sum_{n\in\mathbb{Z}^d} \psi(x - n) \equiv 1.$$

*Let $X$ be a translation invariant Banach space of functions or distributions on $\mathbb{R}^d$ such that $A(\mathbb{R}^d) \cdot X \subseteq X$ with $\|\phi f\|_X \leq \|\phi\|_A \|f\|_X$. The Wiener amalgam space*

$$W(X, \ell^p) = \{f : \|f\|_{W(X,\ell^p)} = (\sum_{n\in\mathbb{Z}^d} \|fT_n\psi\|_X^p)^{\frac{1}{p}} < \infty\}.$$

Since $\{T_n\psi\}$ forms a partition of unity, $f = \sum_{n\in\mathbb{Z}^d} fT_n\psi$. Then the Wiener amalgam norm from Definition 2.4 states that the global decay of $f$ measured via the local $X$-norm of $f$ is similar to that of a $\ell^p$-sequence. The simplest Wiener space is $W = W(L^\infty, \ell^1)$. In fact, $W \cap \mathcal{F}W$ is the largest space on which the Poisson summation formula holds pointwise [Grö01].

Let $\gamma(t) = e^{-\pi|t|^2}$ be the Gaussian function on $\mathbb{R}^d$. Modulation spaces will be defined by introducing a special norm for $f$, that is by imposing a norm on the short-time Fourier transform of $f$.

**Definition 2.5** *The modulation space $M_m^{p,q}(\mathbb{R}^d), 1 \leq p, q \leq \infty$ consists of all tempered distributions $f \in \mathcal{S}(\mathbb{R}^d)$ such that the norm*

$$\|f\|_{M_m^{p,q}} := \left(\int \left(\int |V_\gamma f(x, \omega)|^p m(x, \omega)^p \, dx\right)^{\frac{q}{p}} d\omega\right)^{\frac{1}{q}} \tag{2.6}$$

*is finite. If either $p, q = \infty$, the integral is replaced by the $L^\infty$-norm as usual.*

$M^p_m$ will denote $M^{p,p}_m$ for the sake of shortness. If $m = 1$, we write just $M^p$. For a detailed treatment of the theory of modulation spaces we refer to Chapters 11 and 12 of [Grö01]. In our subsequent analysis we shall be interested in modulation spaces with $p = 1, \infty$. Here we recall only the most important properties of these modulation spaces:

1. $M^{p,q}_m(\mathbb{R}^d)$ is a Banach space [Grö01]: Proposition 11.3.5.

2. The definition of $M^{p,q}_m(\mathbb{R}^d)$ is independent of the choice of $\gamma \in \mathcal{S}(\mathbb{R}^d)$. Different choices of $\gamma$ yield equivalent norms, i.e. for $g_1, g_2 \in \mathcal{S}(\mathbb{R}^d)$, $g_1, g_2 \neq 0$, and $\|f\|_1 := \|V_{g_1} f\|_{L^{p,q}_m}$, $\|f\|_2 := \|V_{g_2} f\|_{L^{p,q}_m}$ there exists $C, C'$, dependent on $g_1, g_2$ such that $C\|f\|_1 \leq \|f\|_2 \leq C'\|f\|_1$ for all $f \in M^{p,q}_m$. [Grö01]: Proposition 11.3.2.

3. $M^1_m(\mathbb{R}^d)$ is invariant under time-frequency shifts (although they need not be isometries!) [Grö01]: Proposition 11.3.5.

4. If $m(\omega, -x) \leq Cm(x, \omega)$, then $M^1_m(\mathbb{R}^d)$ is invariant under the Fourier transform [Grö01]: Proposition 11.3.5.

5. $f \in M^1_m$ implies that $f$ is continuous [Grö01]: Proposition 12.1.4.

6. If $m$ is a polynomial weight, then $\mathcal{S}(\mathbb{R}^d)$ is a dense subset of $M^1_m(\mathbb{R}^d)$ [Grö01]: Proposition 11.3.4.

7. The dual space of $M^1_m(\mathbb{R}^d)$ is $M^\infty_{\frac{1}{m}}(\mathbb{R}^d)$ [Grö01]: Proposition 11.3.6.

Furthermore, we have the following inclusion relations

$$\mathcal{S} \subset M^1_{v_s} \subset M^1 \subset M^\infty \subset M^\infty_{1/v_s} \subset \mathcal{S}'$$

for all polynomial weights $v_s(z) = (1 + |z|)^s$, $s > 0$.

The modulation space $M^1$ has attracted a lot of attention and is now referred to as *Feichtinger's algebra* and denoted $S_0$. It has some additional properties:

1. $M^1$ is a Banach algebra under convolution [Grö01], Proposition 12.1.7:

$$f, g \in M^1 \Rightarrow f * g \in M^1.$$

2. $M^1$ is a Banach algebra under pointwise multiplication [Grö01], Proposition 12.1.7:

$$f, g \in M^1 \Rightarrow f \cdot g \in M^1.$$

3. Let $\mathcal{B}$ be a Banach space of tempered distributions such that $\mathcal{B}$ is invariant under time-frequency shifts and $M^1_m \cap \mathcal{B} \neq \varnothing$. Then $M^1_m$ is embedded in $\mathcal{B}$ [Fei81].

4. Whenever $f \in M^1(\mathbb{R}^d)$, Poisson's summation formula

$$\sum_{k\in\mathbb{Z}^d} f(x+k) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)e^{2\pi ikx} \tag{2.7}$$

holds with absolute convergence of both sums for all $x$.

These properties point out the usefulness of $M^1(\mathbb{R}^d)$: it is a Banach algebra, it is invariant under time-frequency shifts and the Fourier transform, it contains only continuous functions, and is dense in $L^2(\mathbb{R}^d)$.

An alternative definition of the modulation space $S_0$ using Wiener amalgam spaces states that $S_0(\mathbb{R}^d) = W(A(\mathbb{R}^d), \ell^1(\mathbb{Z}^d))$ [FZ98]: Proposition 3.2.6. Here $A(\mathbb{R}^d) = \mathcal{F}L^1(\mathbb{R}^d)$. Thus an equivalent norm on $S_0$ would be induced by any compactly supported $\psi \in A(\mathbb{R}^d)$ with $\sum_{n\in\mathbb{Z}^d} T_n\psi = 1$ almost everywhere and is given by

$$\|f\|_{S_0} := \sum_{n\in\mathbb{Z}^d} \|f \cdot T_n\psi\|_A. \tag{2.8}$$

The dual space $S_0' = M^\infty(\mathbb{R}^d)$ would then coincide with $W(A'(\mathbb{R}^d), \ell^\infty(\mathbb{Z}^d))$. Thus, $S_0'(\mathbb{R}^d)$ contains the Dirac delta function $\delta_0$ and the delta train $\sum_{n\in\mathbb{Z}^d} \delta_n$ [FZ98],[KP06].

Definitions of modulation spaces carry over to sequence spaces $\ell_m^{p,q}(\mathbb{Z}^d)$ after a change in (2.6) to counting measure.

**Definition 2.6** *Let $m$ be a weight function. The weighted mixed-norm sequence space $\ell_m^{p,q}(\mathbb{Z}^{2d})$ is,*

$$\ell_m^{p,q}(\mathbb{Z}^{2d}) = \{\mathbf{c} : (\sum_{l\in\mathbb{Z}^d}(\sum_{k\in\mathbb{Z}^d} |c_{k,l}m(k,l)|^p)^{\frac{q}{p}})^{\frac{1}{q}} < \infty\}$$

## 2.3 Modulation spaces and Gabor analysis

Modulation spaces provide a very suitable class of window functions for Gabor systems. For instance, the Gabor frame operator $S_g$ for $g \in M^1$ is a bounded operator on $L^2(\mathbb{R}^d)$ [Grö01]: 12.1.12, 12.2.1.

The following three statements characterize the extreme usefulness of $M_m^1(\mathbb{R}^d)$ for Gabor frame theory. Let $\Lambda \simeq \mathbb{Z}^{2d}$ be a lattice in $\mathbb{R}^{2d}$.

**Proposition 2.7 ([Grö01]: Proposition 12.2.3-4)** *Let $g \in M_m^1(\mathbb{R}^d), 1 \le p, q \le \infty$. Then the Gabor analysis operator*

$$C_{g,\alpha,\beta} : f \mapsto \{V_g f(\lambda), \lambda \in \Lambda\}$$

*is bounded from $M_m^{p,q}(\mathbb{R}^d)$ into $\ell_m^{p,q}(\mathbb{Z}^{2d})$ for all $\Lambda$. Furthermore, the Gabor synthesis operator*

$$C_g^* : \{c_\lambda\} \mapsto \sum_\lambda c_\lambda \pi(\lambda) g$$

*is bounded from $\ell_m^{p,q}(\mathbb{Z}^{2d})$ into $M_m^{p,q}(\mathbb{R}^d)$. If $p, q < \infty$, the convergence of the sum is unconditional in $M_m^{p,q}(\mathbb{R}^d)$. Otherwise it is weak $*$-convergence in $M_{\frac{1}{m}}^\infty(\mathbb{R}^d)$.*

An obvious consequence of this statement is

**Corollary 2.8 ([Grö01]: Proposition 12.2.5)** *If $g, \gamma \in M^1_m(\mathbb{R}^d)$, then the operator $S_{g,\gamma} : f \mapsto \sum_{\lambda \in \Lambda} V_g f(\lambda)\pi(\lambda)\gamma$ is bounded on $M^{p,q}_m(\mathbb{R}^d)$ for all $1 \le p, q \le \infty$. If $g = \gamma$, then this is the Gabor frame operator $S_g$.*

The $L^2$-frame theory extends naturally to frames for modulation spaces $M^{p,q}_m$, as stated by the following important result about Gabor frames with windows in $M^1_m(\mathbb{R}^d)$,

**Corollary 2.9 ([Grö01]: Proposition 12.2.6)** *If $g, \gamma \in M^1_m(\mathbb{R}^d)$, and $S_{g,\gamma} = I$ on $L^2(\mathbb{R}^d)$. Then*

$$f = \sum_{\lambda \in \Lambda} V_g f(\lambda)\pi(\lambda)\gamma \tag{2.9}$$

*with unconditional convergence in $M^{p,q}_m$ if $1 \le p, q < \infty$. Also there exist $A, B > 0$ such that*

$$A\|f\|_{M^{p,q}_m} \le \|\{V_g f(\lambda) : \lambda \in \Lambda\}\|_{\ell^{p,q}_m} \le B\|f\|_{M^{p,q}_m}. \tag{2.10}$$

Equation (2.9) guarantees the existence of a reconstruction operator, while (2.10) ensures the norm equivalence of functions and their Gabor coefficients. Thus, (2.9) and (2.10) fulfil the requirement for a Banach frame for $M^{p,q}_m(\mathbb{R}^d), 1 \le p, q < \infty$ in the sense of Gröchenig [Grö01].

One important question from Gabor analysis is the quality of the canonical dual window. The $L^2$-theory states nothing more except that the dual window is in $L^2(\mathbb{R}^d)$. A very important question about the properties of the canonical dual window was answered by [GL03]:

**Proposition 2.10 ([GL03])** *Let $g \in M^1_m(\mathbb{R}^d)$ be such that $(g, \Lambda)$ be a frame for $L^2(\mathbb{R}^d)$. Then $S_g$ is invertible on $M^1_m(\mathbb{R}^d)$ and the canonical dual $\tilde{g} = S_g^{-1} g$ belongs also to $M^1_m(\mathbb{R}^d)$.*

## 2.4 Hilbert-Schmidt operators

**Definition 2.11** *A bounded operator $T : \mathcal{H} \to \mathcal{H}$ is called a Hilbert-Schmidt operator if $\sum_{k=1}^{\infty} \|Te_n\|^2_{\mathcal{H}} < \infty$ for some ONB $\{e_n : n \in \mathbb{N}\}$ of $\mathcal{H}$. The Hilbert-Schmidt norm of $T$ is given by the $\ell^2$-norm of the sequence $\{\|Te_n\|_{\mathcal{H}} : n \in \mathbb{N}\}$.*

The Hilbert-Schmidt norm of the operator is independent of the choice of ONB [Con90].

If $H : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is an integral operator with kernel $\kappa_H$, then $H$ is a Hilbert-Schmidt operator if and only if $\kappa_H \in L^2(\mathbb{R}^d)$.

Hilbert-Schmidt operators are integral operators that are completely described by, for example, their *spreading function* $\eta_H$, their *kernel* $\kappa_H$ or their *Kohn-Nirenberg symbol* $\sigma_H$ [Grö01]:

$$\begin{aligned}(Hf)(x) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \eta_H(t,\nu) f(x-t) e^{2\pi i\langle \nu, x-t\rangle} d(t,\nu) \qquad (2.11)\\ &= \int_{\mathbb{R}^d} \kappa_H(x,x-t) f(x-t)\, dt \\ &= \int_{\widehat{\mathbb{R}^d}} \sigma_H(x,\xi)\widehat{f}(\xi) e^{2\pi i\langle x,\xi\rangle} d\xi.\end{aligned}$$

The functions in the above system of equalities are related via Fourier transforms as follows:

$$\eta_H(t,\nu) \underset{\mathcal{F}_{x\mapsto\nu}}{\overset{\mathcal{F}^{-1}_{\nu\mapsto x}}{\rightleftarrows}} \kappa_H(x,x-t) \underset{\mathcal{F}^{-1}_{\xi\mapsto t}}{\overset{\mathcal{F}_{t\mapsto\xi}}{\rightleftarrows}} \sigma_H(x,\xi),$$

or with less compact notation,

$$\begin{aligned}\eta_H(t,\nu) &= \int_{\mathbb{R}^d} \kappa_H(x,x-t) e^{-2\pi i\langle \nu,x\rangle}\, dx = \int_{\mathbb{R}^d\times\mathbb{R}^d} \sigma_H(\xi) e^{-2\pi i(\langle \xi,t\rangle + \langle x,\nu\rangle)}\, d(\xi,x)\\ \kappa_H(x,x-t) &= \int_{\mathbb{R}^d} \eta_H(t,\nu) e^{2\pi i\langle \nu,x\rangle}\, d\nu = \int_{\mathbb{R}^d} \sigma_H(x,\xi) e^{2\pi i\langle \xi,t\rangle}\, d\xi\\ \sigma_H(x,\xi) &= \int_{\mathbb{R}^d} \kappa_H(x,x-t) e^{-2\pi i\langle t,\xi\rangle}\, dt = \int_{\mathbb{R}^d\times\mathbb{R}^d} \eta_H(t,\nu) e^{2\pi i(\langle \xi,t\rangle + \langle x,\nu\rangle)}\, d(t,\nu).\end{aligned}$$

Hilbert-Schmidt operators with spreading functions in $L^2(\mathbb{R}^{2d})$ map $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. In our discussion of the identification problem we want to use distributions as identifiers (in particular, the Dirac delta train), which belong to the modulation space $S_0' = M^\infty(\mathbb{R}^d)$. Thus we will restrict our attention to operators with spreading functions belonging to the Feichtinger algebra $M^1(\mathbb{R}^d)$, or to its proper subspaces, the weighted modulation spaces $M^1_m(\mathbb{R}^d)$. In this case the Hilbert-Schmidt operators extend to $M^\infty(\mathbb{R}^d)$. In our analysis we shall work with operator families belonging to the space of bounded operators $\mathcal{L}(M^\infty, L^\infty)$ [FZ98].

# 3 Identification by solving a linear system

## 3.1 Gabor discretization of Hilbert-Schmidt operators on $L^2(\mathbb{R})$

Hilbert-Schmidt operators on $\mathbb{R}^{2d}$ are exactly the ones for which (2.11) holds with spreading function $\eta_H \in L^2(\mathbb{R}^{2d})$. Hence we can discretize these operators by a Gabor Riesz basis or a Gabor frame decomposition of $\eta_H$.

**Proposition 3.1 (Operator discretization)** *Given that $H_0$ is a Hilbert-Schmidt operator with spreading function $\eta_0$, the operator $T_A M_B T_{-C} H_0 T_C M_D$ has spreading function*

$$\eta_{T_A M_B T_{-C} H_0 T_C M_D} = T_{A,B+D} M_{B,C} \eta_0, \qquad \forall A, B, C, D \in \mathbb{R}^d. \tag{3.1}$$

*In particular, if an operator $H$ has spreading function $\eta_H \in L^2(\mathbb{R}^{2d})$ with Gabor frame series expansion*

$$\eta_H = \sum_{k,l,m,n} c_{k,l;m,n} T_{am,bn} M_{ck,dl} \eta_0 \tag{3.2}$$

*and convergence in $L^2$-norm, then*

$$H = \sum_{k,l,m,n} c_{k,l;m,n} T_{am} M_{ck} T_{-dl} H_0 T_{dl} M_{bn-ck} \overset{\text{def}}{=} \sum_{k,l,m,n} c_{k,l;m,n} H_{k,l;m,n}. \tag{3.3}$$

**Note:** We note first that (3.3) holds with convergence in $L^2(\mathbb{R}^d)$ for sequences $\mathbf{c} = \{c_{k,l;m,n}\} \in \ell^2$. In our study we will be interested in expansions with window $\eta_0$ belonging to the modulation space $M^1_v(\mathbb{R}^d)$, which is a subset of $L^2(\mathbb{R}^d)$.

*Proof.* A change of variables $t := t - A, \nu := \nu - B - D$ and the relation $T_x M_\omega = e^{-2\pi i x\omega} M_\omega T_x$ gives

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} T_{A,B+D} M_{B,C} \eta_0(t,\nu) f(x-t) e^{2\pi i \nu(x-t)}\, d(t,\nu) = \\
&\quad = \iint e^{2\pi i (B(t-A)+C(\nu-B-D))} \eta_0(t-A, \nu-B-D) f(x-t) e^{2\pi i\nu(x-t)}\, d(t,\nu) \\
&\quad = \iint e^{2\pi i(Bt+C\nu)} \eta_0(t,\nu) f(x-t-A) e^{2\pi i(\nu+B+D)(x-t-A)}\, d(t,\nu) \\
&\quad = \iint e^{2\pi i(Bt+C\nu)} \eta_0(t,\nu) T_{t+A} M_{\nu+B+D} f(x)\, d(t,\nu) \\
&\quad = T_A M_B T_{-C} \iint \eta_0(t,\nu) T_t M_\nu T_C M_D f(x)\, d(t,\nu) \\
&\quad = T_A M_B T_{-C} H_0 T_C M_D f(x).
\end{aligned}$$

Hence, in particular,

$$\begin{aligned}
(Hf)(x) &= \iint_{\mathbb{R}^{2d}} \eta_H(t,\nu) f(x-t) e^{2\pi i\nu(x-t)}\, d(t,\nu) \\
&= \sum_{k,l,m,n} c_{k,l;m,n} \iint T_{am,bn} M_{ck,dl} \eta_0(t,\nu) f(x-t) e^{2\pi i\nu(x-t)}\, d(t,\nu) \\
&= \sum_{k,l,m,n} c_{k,l;m,n} T_{am} M_{ck} T_{-dl} H_0 T_{dl} M_{bn-ck} f(x).
\end{aligned}$$

□

Expansions based on different time-frequency lattices are listed in Table 2. Expansion (3.2) corresponding to the most general diagonal matrix corresponds to Case A from this Table. Case B2 has been solved in [KP05] for bandlimited $\eta_0 \in S_0(\mathbb{R})$.

## 3.2 Solving the linear system

With $H = \sum_{k,l,m,n} c_{k,l;m,n} H_{k,l;m,n}$ as in (3.3), the *identification* problem is to find a function $f \in L^2$ such that any choice of coefficients $c_{k,l;m,n}$ from (some subclass of) $\ell^2(\mathbb{Z}^{4d})$, can be computed from $Hf$, or equivalently, from the coefficients $v_{i,j} = \langle Hf, M_{pi}T_{qj}\gamma\rangle$ of a Gabor frame series expansion with window $\gamma$ and lattice constants $p, q$. Hence, we need to solve the equation

$$\begin{aligned} v_{i,j} =& \langle Hf, M_{pi}T_{qj}\gamma\rangle = \sum_{k,l,m,n} c_{k,l;m,n}\langle H_{k,l;m,n}f, M_{pi}T_{qj}\gamma\rangle \\ =& \sum_{k,l,m,n} c_{k,l;m,n}A_{i,j;k,l,m,n}. \end{aligned} \tag{3.4}$$

In other words, we need to choose $f$ so that the mapping $A\colon \ell^2(\mathbb{Z}^{4d}) \to \ell^2(\mathbb{Z}^{2d})$ with $\mathbb{Z}^{2d} \times \mathbb{Z}^{4d}$-matrix representation $A = (A_{i,j;k,l,m,n})$ has an inverse, i.e. $A$ is invertible at least on some well-defined subset of $\ell^2(\mathbb{Z}^{4d})$.

The following lemma will be used for our computations.

**Lemma 3.2** *Let $\eta_H, f, g \in L^2(\mathbb{R}^d)$. Then*

$$\langle Hf, g\rangle = \langle \eta_H, V_f g\rangle_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \tag{3.5}$$

*Proof.* Fubini applied to (2.11) gives

$$\begin{aligned} \langle Hf, g\rangle =& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d\times\mathbb{R}^d} \eta_H(t,\nu) f(x-t) e^{2\pi i\langle \nu, x-t\rangle}\, d(t,\nu)\overline{g(x)}\, dx \\ =& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_H(t,\nu) \int_{\mathbb{R}^d} f(x-t) e^{2\pi i\langle \nu, x-t\rangle}\overline{g(x)}\, dx\, dt\, d\nu \\ =& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_H(t,\nu) \overline{\int_{\mathbb{R}^d} \overline{f(x-t)} e^{-2\pi i\langle \nu, x-t\rangle} g(x)\, dx}\, dt\, d\nu \\ =& \langle \eta_H, V_f g\rangle_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \end{aligned}$$

We note that if $\eta_H \in S_0(\mathbb{R}^d)$ and has compact support, then (3.5) holds for all $f \in S_0'(\mathbb{R}^d)$ because $V_g f \in L^\infty(\mathbb{R}^{2d})$. Q.E.D. □

Now we return to the problem of discretization. We do the discretization in two steps. First we do a parametrization (3.6) of $f$ in Lemma 3.3. Then, in a list of examples, summarized in Table 2 and Table 1, we consider different lattices and subclasses of spreading functions and show in some cases how the coefficients $a_{r,s}$ in (3.7) can be chosen so that the mapping $A$ is invertible (if this is possible).

The following lemma shows how the matrix coefficients of $A$ can be computed from the Gabor frame coefficients of $f$:

**Lemma 3.3** *Suppose that*

$$f = \sum_{r,s\in\mathbb{Z}^d} a_{r,s} M_{\alpha r} T_{\beta s}\tilde{g}. \tag{3.6}$$

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| $\eta_0$<br>$H_0$ | $\delta_0(t)\delta_0(\nu)$<br>$f \mapsto f$ | $p(t)\delta_0(\nu)$<br>$f \mapsto p * f$ | $\delta_0(t)q(\nu)$<br>$f \mapsto f \cdot \overline{\widehat{q}}$ | $p(t)q(\nu)$<br>$f \mapsto p * (f \cdot \overline{\widehat{q}})$ | $V_{g_1}g_2(t,\nu)$<br>$f \mapsto g_2\langle f, g_1\rangle$ |
| A,A' | $D_{f,\alpha,\gamma}c$<br>$(n,l=0)$ | $\sum c_{k,l,m} M_{\beta l} T_{\alpha k} V_{p^*} f(x,-\beta m)$<br>$(n=0)$ | $\sum c_{k,m,n} T_{\alpha k} M_{\beta m}(\overline{\widehat{q}} \cdot T_{\alpha n} f)$<br>$(l=0)$ | | |
| B1 | $D_{f,\alpha,\gamma}c$<br>*synthesis operator* | $\sum c_{k,m}(T_{\alpha k} p * M_{\gamma m} f)$ | $D_{f\cdot\overline{\widehat{q}},\alpha,\gamma}c$<br>synthesis | $\sum c_{k,m} V_{p^*}(\overline{\widehat{q}} f)(x-\alpha k, -\gamma m)$ | $\sum c_{k,m}\langle f, M_{\gamma m} g_1\rangle T_{\alpha k} g_2$ |
| B2 | $f\langle c, \delta_{\mathbb{Z}^2}\rangle$ | $f * (\sum c'_l M_{\beta l} p)$ | $f \cdot (\sum c'_n T_{-\delta n} \overline{\widehat{q}})$ | $\sum c_{l,n}(M_{\beta l} p) * ((T_{\delta n} \overline{\widehat{q}}) f)$ | $\sum c_{l,n} V_{g_1} f(-\delta n, \beta l) M_{\beta l} T_{-\delta n} g_2$<br>*Gabor multiplier* |
| D1 | $D_{f,\alpha,\beta}c$<br>synthesis | $\sum c_{k,m}(T_{\alpha k} p * M_{\beta m} f)$ | $\sum c'_{m,n}(M_{\beta m} \overline{\widehat{q}} \cdot T_{\alpha n} f)$ | $p * (\sum c'_{m,n}(M_{\beta m} \overline{\widehat{q}} \cdot T_{\alpha n} f))$ | $\langle c, C_{g_1,\alpha,\beta} f\rangle_{\ell^2} g_2$ |
| E1 | $D_{f,\alpha,\beta}c$<br>synthesis | $D_{p*f,\alpha,\beta}c$<br>synthesis | $D_{\overline{\widehat{q}}\cdot f,\alpha,\beta}c$<br>synthesis | $D_{p*(\overline{\widehat{q}}\cdot f),\alpha,\beta}c$<br>synthesis | $\langle f, g_1\rangle D_{g_2,\alpha,\beta}c$ |
| F1 | $D_{f,\alpha,\beta}c$ | $\sum c_{k,l} M_{\beta l}(p * T_{\alpha k} f)$ | $\sum c_{k,l}(M_{\beta l} \overline{\widehat{q}} \cdot T_{\alpha k} f)$ | $\sum c_{k,l}(M_{\beta l} p * \overline{\widehat{q}} \cdot T_{\alpha k} f)$ | $\sum c_{k,l}\langle f, T_{-\alpha k} g_1\rangle M_{\beta l} g_2$ |

Table 1: Special generators $\eta_0$.

*Then the matrix elements in* (3.4) *are*

$$A_{i,j;k,l,m,n} = \sum_{r,s\in\mathbb{Z}^d} a_{r,s} e^{2\pi i(\langle \beta s, bn-pi+\alpha r\rangle - \langle pi, ma\rangle)} \times \qquad\qquad \langle T_{ma-qj+\beta s, bn-pi+\alpha r} M_{ck-pi, ld+\beta s}\eta_0, V_{\tilde{g}}\gamma\rangle_{L^2(\mathbb{R}^{2d})} \tag{3.7}$$

*Proof.* Since $f = \sum_{r,s\in\mathbb{Z}^d} a_{r,s} M_{\alpha r} T_{\beta s}\tilde{g}$, we know from (3.4) that

$$\begin{aligned}
A_{i,j;k,l,m,n} =& \langle H_{k,l;m,n} f, M_{pi}T_{qj}\gamma\rangle \\
=& \sum_{r,s\in\mathbb{Z}^d} a_{r,s}\langle H_{k,l;m,n} M_{\alpha r} T_{\beta s}\tilde{g}, M_{pi}T_{qj}\gamma\rangle \\
=& \sum_{r,s\in\mathbb{Z}^d} a_{r,s}\langle T_{am}M_{ck}T_{-dl}H_0 T_{dl} M_{bn-ck} M_{\alpha r} T_{\beta s}\tilde{g} M_{pi}T_{qj}\gamma\rangle \\
=& \sum_{r,s\in\mathbb{Z}^d} a_{r,s}\langle T_{-qj}M_{-pi}T_{am}M_{ck}T_{-dl}H_0 T_{dl} M_{bn-ck} M_{\alpha r} T_{\beta s}\tilde{g}, \gamma\rangle \\
=& \sum_{r,s\in\mathbb{Z}^d} a_{r,s} e^{2\pi i(-\langle pi, am\rangle + \langle \beta s, ck-pi+bn-ck+\alpha r\rangle)} \times \\
& \times \langle T_{-qj+am+\beta s} M_{-pi+ck} T_{-dl-\beta s} H_0 T_{dl+\beta s} M_{bn-ck+\alpha r}\tilde{g}, \gamma\rangle .
\end{aligned}$$

In combination with (3.5) and (3.1), this gives (3.7). □

After [KP06] we know that a certain class of operators with compactly supported spreading function is identifiable.

We illustrate the advantage of the method of solving a linear system by giving an alternative proof of a result from [KP06].

**Theorem 3.4 ([KP06])** *The set of operators* $\mathcal{H} = \{H : \operatorname{supp}\eta_H \subseteq [0,a) \times [0,\frac{1}{a}), \eta_H \in M^1\}$ *is identifiable.*

*Proof.* The study [KP06] considers the operator family $\mathcal{H} = \{H : \operatorname{supp}\eta_H \subseteq [0,a) \times [0,\frac{1}{a}), \eta_H \in M^1\}$. This operator family can be alternatively defined as consisting of operators $H$ such that

$$\eta_H \in M^1 \cap \overline{\operatorname{span}}\,\{\pi(\lambda)\chi_{[0,a)\times[0,\frac{1}{a})}, \lambda \in \Lambda\},$$

for

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{a} \\ 0 & 0 & a & 0 \end{pmatrix}^T \mathbb{Z}^{2d},$$

as defined in this theorem.

It is well known that the family of exponentials $\{e^{2\pi i(\frac{mx}{a}+any)} : (x,y) \in [0,a) \times [0,\frac{1}{a}), m,n \in \mathbb{Z}\}$ form an orthonormal basis for the space $L^2([0,a)\times[0,\frac{1}{a}))$. Therefore, if we let $\eta_0 = \chi_{[0,a]\times[0,\frac{1}{a}]}$, we obtain the following expansion of the spreading function of the operator $H$,

$$\eta_H = \sum_{k,l} c_{k,l} M_{\frac{k}{a},al}\eta_0$$

We can set thus $c_{k,l,m,n} = c_{k,l}\delta_{m,n}(0,0)$. Furthermore, we set $\gamma = \chi_{[0,a)}$ and use a Gabor frame $(\gamma, a\mathbb{Z}\times\frac{1}{a}\mathbb{Z})$ for a resolution of the coefficients $v_{i,j} = \langle Hf, M_{\frac{i}{a}}T_{aj}\gamma\rangle$. We employ as identifier the Dirac delta-train $f = \tilde{g} = \delta_{a\mathbb{Z}}$, which belongs to $\mathcal{S}'(\mathbb{R})$.

We use a canonical representation of the identifier $f$ with $a_{r,s} = \delta_{0,0}(r,s)$. After substitution in (3.7) (note here that $\eta_0$ has compact support, so the inner product (3.5) is still well-defined even though $V_{\delta_{a\mathbb{Z}}}\gamma \notin L^2(\mathbb{R}^2)$) we obtain

$$A_{i,j;k,l} = \langle T_{-aj,-\frac{i}{a}} M_{\frac{k}{a}-\frac{i}{a},al}\eta_0, V_{\delta_{a\mathbb{Z}}}\gamma\rangle,$$

which we rewrite as follows: On one hand we use the following property of the short-time Fourier transform $V_{\delta_{a\mathbb{Z}}}\gamma = Z_a\gamma$. This follows from the fact that $V_{\delta_{a\mathbb{Z}}}g = Z_a g$ for $g \in S_0(\mathbb{R}^d)$, which can be extended by density for all $g \in L^2(\mathbb{R}^d)$. On the other hand, $Z_a\chi_{[0,a)}(x,\omega) = e^{2\pi i a\left[\frac{x}{a}\right]\omega}$. Rewriting $\eta_0$ into a tensor product $\eta_0 = \chi_{[0,a)} \otimes \chi_{[0,\frac{1}{a})}$ will allow us to compute the integral of the inner product:

$$A_{i,j;k,l} = \iint \chi_{[0,a)}(x+aq)e^{2\pi i\frac{(k-p)(x+aq)}{a}}\chi_{[0,\frac{1}{a})}(\omega+\tfrac{i}{a})e^{2\pi i\left(\omega+\frac{i}{a}\right)al}e^{-2\pi i a\left[\frac{x}{a}\right]\omega}dx d\omega$$

We make subsitution $y = x + aj$, $\xi = \omega + \frac{i}{a}$ and note that since the integrand is nonzero for $aj \le x < aj + a$, $\left[\frac{x}{a}\right] = aj$.

$$\begin{aligned}A_{i,j;k,l} =& \int_0^a \int_0^{\frac{1}{a}} e^{2\pi i\frac{(k-i)y}{a}} e^{2\pi i(\frac{a}{l}\xi - 2\pi i aj(\xi-\frac{i}{a})} dy d\xi \\ =& e^{2\pi i(ij)} \int_0^a e^{2\pi i\frac{(k-i)y}{a}} dy \times \int_0^{\frac{1}{a}} e^{2\pi i a(l-j)\xi} d\xi \\ =& \delta_{i,j}(k,l),\end{aligned}$$

because the families of exponentials $\{e^{2\pi i\frac{n}{a}y} : n \in \mathbb{Z}\}$ and $\{e^{2\pi i may} : m \in \mathbb{Z}\}$ form an orthonormal basis for $L^2[0,a)$ and $L^2[0,\frac{1}{a})$ respectively.

The matrix $A = A_{i,j;k,l}$ is diagonal, and moreover, it is the identity, and thus, the coefficients of the expansion of $\eta_0$ are equal to the coefficients of the frame expansion $c_{k,l} = v_{k,l} = \langle H\delta_{a\mathbb{Z}}, M_{\frac{k}{a}}T_{al}\chi_{[0,a)}\rangle$. □

Case A from Table 2 shows more concretely the series expansion of the mapping $f \mapsto Hf$ in terms of the Gabor Riesz coefficients of $\eta_H$.

# 4 Operator families associated with general lattices

In the general case we will consider a collection of operators whose spreading functions have a set, pre-determined Riesz basis expansion with respect to some Gabor system or a subset of a Gabor system. Namely, we will look at

$$\mathcal{H} = \{H : \eta_H \in \overline{\text{span}}\{\pi(\lambda)\eta_0 : \lambda \in \Lambda\}\},$$

| Ref. | Lattice | Frame elements | Prototype operator |
| --- | --- | --- | --- |
| A | $\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \mathbb{Z}^4$ | $T_{\alpha k,\beta l} M_{\gamma m,\delta n} \eta_0$ | $T_{\alpha k} M_{\gamma m} T_{-\delta n} H_0 T_{\delta n} M_{\beta l-\gamma m}$ |
| B1 | $\begin{pmatrix} 0 & \gamma & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\gamma l} \eta_0$ | $T_{\alpha k} H_0 M_{\gamma l}$ |
| B2 | $\begin{pmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & \alpha & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $M_{\alpha k,\beta l} \eta_0$ | $M_{\alpha k} T_{-\beta l} H_0 T_{\beta l} M_{-\alpha k}$ |
| B3 | $\begin{pmatrix} 0 & 0 & 0 & \beta \\ \alpha & 0 & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,0} M_{0,\beta l} \eta_0$ | $T_{\alpha k-\beta l} H_0 T_{\beta l}$ |
| B4 | $\begin{pmatrix} 0 & 0 & \beta & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{0,\alpha k} M_{\beta l,0} \eta_0$ | $M_{\beta l} H_0 M_{\alpha k-\beta l}$ |
| B5 | $\begin{pmatrix} 0 & 0 & 0 & \beta \\ 0 & \alpha & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{0,\alpha k} M_{0,\beta l} \eta_0$ | $T_{-\beta l} H_0 T_{\beta l} M_{\alpha k}$ |
| B6 | $\begin{pmatrix} 0 & 0 & \beta & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,0} M_{\beta l,0} \eta_0$ | $T_{\alpha k} M_{\beta l} H_0 M_{-\beta l}$ |
| D1 | $\begin{pmatrix} 0 & \beta & 0 & 0 \\ \alpha & 0 & 0 & \alpha \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l} M_{0,\alpha k} \eta_0$ | $H_0 T_{\alpha k} M_{\beta l}$ |
| D2 | $\begin{pmatrix} 0 & 0 & 0 & \alpha \\ \alpha & \beta & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta k} M_{0,\alpha l} \eta_0$ | $T_{\alpha(k-l)} H_0 T_{\alpha l} M_{\beta k}$ |
| D3 | $\begin{pmatrix} 0 & \beta & 0 & \alpha \\ \alpha & 0 & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l} M_{0,\alpha l} \eta_0$ | $T_{\alpha(k-l)} H_0 T_{\alpha l} M_{\beta l}$ |
| D4 | $\begin{pmatrix} 0 & \beta & 0 & 0 \\ \alpha & 0 & \alpha & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l} M_{\alpha k,0} \eta_0$ | $T_{\alpha k} M_{\alpha k} H_0 M_{\beta l-\alpha k}$ |
| D5 | $\begin{pmatrix} 0 & 0 & \alpha & 0 \\ \alpha & \beta & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta k} M_{\alpha l,0} \eta_0$ | $T_{\alpha k} M_{\alpha l} H_0 M_{\beta k-\alpha l}$ |
| D6 | $\begin{pmatrix} 0 & \beta & \alpha & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l} M_{\alpha l,0} \eta_0$ | $T_{\alpha k} M_{\alpha l} H_0 M_{(\beta-\alpha)l}$ |
| E1 | $\begin{pmatrix} 0 & \beta & \beta & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l} M_{\beta l,0} \eta_0$ | $T_{\alpha k} M_{\beta l} H_0$ |
| E2 | $\begin{pmatrix} 0 & 0 & \beta & 0 \\ \alpha & \beta & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta k} M_{\beta l,0} \eta_0$ | $T_{\alpha k} M_{\beta l} H_0 M_{\beta(k-l)}$ |
| E3 | $\begin{pmatrix} 0 & \beta & 0 & 0 \\ \alpha & 0 & \beta & 0 \end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l} M_{\beta k,0} \eta_0$ | $T_{\alpha k} M_{\beta k} H_0 M_{\beta(l-k)}$ |

Table 2: Expansions of operators based on Gabor Riesz expansions with window $\eta_0$ of the spreading function $\eta_H$.

| Ref. | Lattice | Frame elements | Prototype operator |
|---|---|---|---|
| E4 | $\begin{pmatrix}0&\beta&0&\beta\\ \alpha&0&0&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l}M_{0,\beta l}\eta_0$ | $T_{\alpha k-\beta l}H_0T_{\beta l}M_{\beta l}$ |
| E5 | $\begin{pmatrix}0&0&0&\beta\\ \alpha&\beta&0&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta k}M_{0,\beta l}\eta_0$ | $T_{\alpha k-\beta l}H_0T_{\beta l}M_{\beta k}$ |
| E6 | $\begin{pmatrix}0&\beta&0&0\\ \alpha&0&0&\beta\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l}M_{0,\beta k}\eta_0$ | $T_{(\alpha-\beta)k}H_0T_{\beta k}M_{\beta l}$ |
| F1 | $\begin{pmatrix}0&\beta&\beta&0\\ \alpha&0&\alpha&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l}M_{\beta l,\alpha k}\eta_0$ | $e^{2\pi i\alpha k\cdot\beta l}M_{\beta l}H_0T_{\alpha k}$ |
| F2 | $\begin{pmatrix}0&\beta&0&\alpha\\ \alpha&0&\beta&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l}M_{\beta k,\alpha l}\eta_0$ | $e^{-2\pi i\beta k\cdot\alpha k}M_{\beta k}T_{\alpha(k-l)}H_0T_{\alpha l}M_{\beta(l-k)}$ |
| F3 | $\begin{pmatrix}0&0&\beta&\alpha\\ \alpha&\beta&0&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta k}M_{\beta l,\alpha l}\eta_0$ | $e^{-2\pi i\beta l\cdot\alpha k}M_{\beta l}T_{\alpha(k-l)}H_0T_{\alpha l}M_{\beta(k-l)}$ |
| F4 | $\begin{pmatrix}0&\beta&0&\beta\\ \alpha&0&\alpha&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l}M_{\alpha k,\beta l}\eta_0$ | $e^{2\pi i\alpha k\cdot\beta l}T_{\alpha k-\beta l}M_{\alpha k}H_0T_{\beta k}M_{\beta l-\alpha k}$ |
| F5 | $\begin{pmatrix}0&\beta&\alpha&0\\ \alpha&0&0&\beta\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta l}M_{\alpha l,\beta k}\eta_0$ | $e^{2\pi i\beta k\cdot\alpha l}T_{(\alpha-\beta)k}M_{\alpha m}H_0T_{\beta k}M_{(\beta-\alpha)m}$ |
| F6 | $\begin{pmatrix}0&0&\alpha&\beta\\ \alpha&\beta&0&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,\beta k}M_{\alpha l,\beta l}\eta_0$ | $e^{2\pi i\beta l\cdot\alpha l}T_{\alpha k-\beta l}M_{\alpha l}H_0T_{\beta l}M_{\beta k-\alpha l}$ |
| G1 | $\begin{pmatrix}0&0&0&\alpha\\ 0&\beta&\beta&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{0,\beta l}M_{\beta l,\alpha n}\eta_0$ | $M_{\beta l}T_{-\alpha n}H_0T_{\alpha n}$ |
| G2 | $\begin{pmatrix}0&0&\beta&\alpha\\ 0&\beta&0&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{0,\beta m}M_{\beta l,\alpha l}\eta_0$ | $M_{\beta l}T_{-\alpha l}H_0T_{\alpha l}M_{\beta(m-l)}$ |
| G3 | $\begin{pmatrix}0&0&\beta&0\\ 0&\beta&0&\alpha\end{pmatrix}^T \mathbb{Z}^2$ | $T_{0,\beta m}M_{\beta l,\alpha m}\eta_0$ | $M_{\beta l}T_{-\alpha m}H_0T_{\alpha m}M_{\beta(m-l)}$ |
| G4 | $\begin{pmatrix}0&0&\alpha&0\\ 0&\beta&0&\beta\end{pmatrix}^T \mathbb{Z}^2$ | $T_{0,\beta l}M_{\alpha n,\beta l}\eta_0$ | $M_{\alpha n}T_{-\beta l}H_0T_{\beta l}M_{\beta l-\alpha n}$ |
| G5 | $\begin{pmatrix}0&0&\alpha&\beta\\ 0&\beta&0&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{0,\beta m}M_{\alpha l,\beta l}\eta_0$ | $M_{\alpha l}T_{-\beta l}H_0T_{\beta l}M_{\beta m-\alpha l}$ |
| G6 | $\begin{pmatrix}0&0&0&\beta\\ 0&\beta&\alpha&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{0,\beta m}M_{\alpha m,\beta l}\eta_0$ | $M_{\alpha m}T_{-\beta l}H_0T_{\beta l}M_{(\beta-\alpha)m}$ |
| H1 | $\begin{pmatrix}0&0&\beta&\alpha\\ \alpha&0&0&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,0}M_{\beta l,\alpha l}\eta_0$ | $e^{2\pi i\alpha l\beta l}T_{\alpha(k-l)}M_{\beta l}H_0T_{\alpha l}M_{-\beta l}$ |
| H2 | $\begin{pmatrix}0&0&0&\alpha\\ \alpha&0&\beta&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,0}M_{\beta k,\alpha l}\eta_0$ | $e^{2\pi i\alpha l\beta k}T_{\alpha(k-l)}M_{\beta k}H_0T_{\alpha l}M_{-\beta k}$ |
| H3 | $\begin{pmatrix}0&0&\beta&0\\ \alpha&0&0&\alpha\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,0}M_{\beta l,\alpha k}\eta_0$ | $M_{\beta l}H_0T_{-\alpha k}M_{-\beta l}$ |
| H4 | $\begin{pmatrix}0&0&\alpha&\beta\\ \alpha&0&0&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,0}M_{\alpha l,\beta l}\eta_0$ | $e^{2\pi i\alpha l\beta l}T_{\alpha k-\beta l}M_{\alpha l}H_0T_{\beta l}M_{-\alpha l}$ |
| H5 | $\begin{pmatrix}0&0&\alpha&0\\ \alpha&0&0&\beta\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,0}M_{\alpha n,\beta k}\eta_0$ | $e^{2\pi i\alpha n\beta k}T_{(\alpha-\beta)k}M_{\alpha n}H_0T_{\beta k}M_{-\alpha n}$ |
| H6 | $\begin{pmatrix}0&0&0&\beta\\ \alpha&0&\alpha&0\end{pmatrix}^T \mathbb{Z}^2$ | $T_{\alpha k,0}M_{\alpha k,\beta l}\eta_0$ | $e^{2\pi i\alpha k\beta l}T_{\alpha k-\beta l}M_{\alpha k}H_0T_{\beta l}M_{-\alpha k}$ |



Table 3: Expansions of operators based on Gabor Riesz expansions with window $\eta_0$ of the spreading function $\eta_H$.

with $\Lambda$ a lattice in $\mathbb{R}^{4d}$, $\eta_0$ a window function in some fixed space. Due to the form of equation (3.4), we see that the index set given by $\Lambda$ must be determined by two indices in order for the problem to be well-defined (dimension-counting). Otherwise, the map defined by (3.4) sends variables from a space with 4 degrees of freedom to variables with just 2 degrees of freedom. In particular we will work with special $\Lambda$, namely a regularly spaced point set within a $2d$-dimensional subspace of $\mathbb{R}^{4d}$. Furthermore we shall examine identifiability of the respective operator class depending on some measure of density of $\Lambda$ as measured within the copy of $\mathbb{R}^{2d}$ it lies in.

Before we define the particular type of density to be used in the subsequent analysis, we recall the definition of Beurling density. Let $B_d(R)$ denote a ball in $\mathbb{R}^d$ centered at 0 with radius $R$.

**Definition 4.1** *Let $\Lambda$ be a set of points inside $\mathbb{R}^d$. Then the lower and upper Beurling densities of $\Lambda$ are given by*

$$\begin{aligned} D^-(\Lambda) &= \liminf_{R\to\infty} \inf_{z\in\mathbb{R}^d} \frac{|\Lambda \cap \{B_d(R)+z\}|}{\pi R^d}, \\ D^+(\Lambda) &= \limsup_{R\to\infty} \sup_{z\in\mathbb{R}^d} \frac{|\Lambda \cap \{B_d(R)+z\}|}{\pi R^d}. \end{aligned} \tag{4.1}$$

*Whenever $D^-(\Lambda) = D^+(\Lambda)$, we speak of the Beurling density of $\Lambda$, $D(\Lambda)$.*

Clearly, whenever $\Lambda$ is a lattice, the Beurling density is the inverse of the area of the fundamental domain of $\Lambda$.

For the purposes of studying operator families $\mathcal{H}_\Lambda$, where $\Lambda$ is a point set lying within a $2d$-dimensional hyperplane $\mathbb{S} \subset \mathbb{R}^{4d}$ we modify the definition of density. The combinations listed in Table 2 and Table 3 correspond to standard $2d$-dimensional lattices in standard hyperplanes inside $\mathbb{R}^{4d}$.

**Definition 4.2** *Let $\Lambda$ be a point set lying inside a $2d$-dimensional subspace of $\mathbb{R}^{4d}$, denoted by $\mathbb{S}$. The "$2d$-dimensional" Beurling densities (or for short $2d$-density) of $\Lambda$ are given by*

$$\begin{aligned} D^-_{(2)}(\Lambda) &= \liminf_{R\to\infty} \inf_{z\in\mathbb{S}} \frac{|\Lambda \cap \{B_{4d}(R)+z\}|}{\pi R^{2d}}, \\ D^+_{(2)}(\Lambda) &= \limsup_{R\to\infty} \sup_{z\in\mathbb{S}} \frac{|\Lambda \cap \{B_{4d}(R)+z\}|}{\pi R^{2d}} \end{aligned} \tag{4.2}$$

We point out that in contrast to Definition 4.1 of the classical Beurling density, the denominators in (4.2) contain the power $R^{2d}$, which corresponds to the dimension of $\mathbb{S} \supset \Lambda$. If the classical Beurling density (Definition 4.1) were used, the densities of $\mathbb{S}$ and of $\Lambda$ would equal 0.

First of all in our analysis we consider a time-frequency lattice $\Lambda$ parametrized by a $4 \times 2$-matrix. We shall compute a formula for the 2-denisty of $\Lambda$ as defined by (4.2). Suppose we have the following parametrization:

$$\Lambda = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} \mathbb{Z}^2 = \left\{ \begin{pmatrix} a_1 m + a_2 n \\ b_1 m + b_2 n \\ c_1 m + c_2 n \\ d_1 m + d_2 n \end{pmatrix} : m, n \in \mathbb{Z} \right\} \tag{4.3}$$

This is a two-dimensional set of points in $\mathbb{R}^4$. The 2-density will depend on all four pairs of coefficients of the matrix.

In accordance with the formulae (4.2), the "2-dimensional" lower and upper Beurling densities of $\Lambda$ are given by

$$D^-_{(2)}(\Lambda) = \liminf_{R\to\infty} \inf_{z\in\mathbb{S}} \frac{|\Lambda \cap \{B_4(R) + z\}|}{\pi R^2}, \quad D^+_{(2)}(\Lambda) = \limsup_{R\to\infty} \sup_{z\in\mathbb{S}} \frac{|\Lambda \cap \{B_4(R) + z\}|}{\pi R^2}$$

The numerators are a count of the number of points of $\mathbb{Z}^2$ that lie within the ellipse $E_\Lambda^{(R)} \subset \mathbb{S}$ defined by the inequality

$$\mathcal{E}_\Lambda^{(R)} : (a_1 m + a_2 n)^2 + (b_1 m + b_2 n)^2 + (c_1 m + c_2 n)^2 + (d_1 m + d_2 n)^2 \leq R^2.$$

When $\Lambda$ is parametrized by a matrix, it is regularly spaced inside the hyperplane, so when $R \to \infty$, the expressions $D^-_{(2)}(\Lambda)$ and $D^+_{(2)}(\Lambda)$ converge to

$$D_{(2)}(\Lambda) = \frac{m(\mathcal{E}_\Lambda)}{\pi},$$

where $m(\mathcal{E}_\Lambda)$ is area of the ellipse $\mathcal{E}_\Lambda$. To find this area we rewrite the equation of the ellipse as follows:

$$\begin{aligned} \mathcal{E}_\Lambda : \quad & (a_1 m + a_2 n)^2 + (b_1 m + b_2 n)^2 \\ & \quad + (c_1 m + c_2 n)^2 + (d_1 m + d_2 n)^2 \leq 1 \Leftrightarrow \\ \mathcal{E}_\Lambda : \quad & (a_1^2 + b_1^2 + c_1^2 + d_1^2)m^2 + (a_2^2 + b_2^2 + c_2^2 + d_2^2)n^2 \\ & \quad + 2(a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2)mn \leq 1 \end{aligned}$$

The area of $\mathcal{E}_\Lambda$ is computed to be

$$m(\mathcal{E}_\Lambda) = \frac{\pi}{\sqrt{w_\Lambda}},$$

where

$$\begin{aligned} w_\Lambda = & (a_1 b_2 - a_2 b_1)^2 + (a_1 c_2 - a_2 c_1)^2 + (a_1 d_2 - a_2 d_1)^2 \\ & + (b_1 c_2 - b_2 c_1)^2 + (b_1 d_2 - b_2 d_1)^2 + (c_1 d_2 - c_2 d_1)^2 \end{aligned}$$

Hence, we obtain that the '2-dimensional density' of $\Lambda$ equals

$$D_{(2)}(\Lambda) = \frac{1}{\sqrt{w_\Lambda}}.$$

In the following we will focuses our attention at some of the cases illustrated in Figures 2 and 3 under assumptions on $\eta_0$ belonging to the modulation space $M^1_v(\mathbb{R}^2)$, for a polynomial weight $v(z) = (1 + |z|)^n, n > 2$. We shall work with operator classes where the spreading functions are given by Riesz basic expansions with respect to a Gabor system $(\eta_0, \Lambda)$, where $\eta_0 \in M^1_v(\mathbb{R}^2)$ and $\Lambda$ is a set of points parametrized by $\mathbb{Z}^2$ (a regular lattice lying in some plane within $\mathbb{R}^4$ - different choices of parameters are listed in Table 2 and 3). The primary goal is to formulate results on a relation between identifiability and non-identifiability and the measure of the *2-density* of $\Lambda$.

## 4.1 Necessity of the Riesz sequence condition on $\{\pi(\lambda)\eta_0 : \lambda \in \Lambda\}$

The following example illustrates the fact that by discretizing the operator in terms of a Gabor expansion of its spreading function, we identify 'coefficients' and not the operator itself. The fact that in our approach we work not with the operators themselves, but with the coefficients of their Gabor representations, requires uniqueness of the coefficient representation. In other words, to the zero operator we must in our framework associate only the zero sequence. Therefore, the Riesz sequence condition is necessary for the method of our analysis of identification.

Our example demonstrates what happens if $\{M_{\alpha k, \beta l}\eta_0 : k, l \in \mathbb{Z}^d\}$ is not a Riesz sequence in $L^2$. Consider the collection of operators $\mathcal{H} = \{H : \operatorname{supp} \eta_H \subset [-\frac{1}{4}, \frac{1}{4})^2\}$.

It is well known that the family $\{M_{k,l}\eta_0 : k, l \in \mathbb{Z}\}$, where $\eta_0 = \chi_{[-\frac{1}{4},\frac{1}{4})\times[-\frac{1}{4},\frac{1}{4})}$, is not a Riesz sequence. To see this, it is enough to show that in the one-dimensional case $\{M_k \chi_{[-\frac{1}{4},\frac{1}{4})} : k \in \mathbb{Z}\}$ is not a Riesz basis for $L^2[-\frac{1}{4}, \frac{1}{4})$. The system

$$\{\chi_{[-\frac{1}{4},\frac{1}{4})}(t)e^{2\pi i(2n)t} : n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2[-\frac{1}{4}, \frac{1}{4})$. Also

$$\{\chi_{[-\frac{1}{4},\frac{1}{4})}(t)e^{2\pi i(2n+1)t} : n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2[-\frac{1}{4}, \frac{1}{4})$. We denote for sake of clarity, $e_n = \chi_{[-\frac{1}{4},\frac{1}{4})}(t)e^{2\pi int}$. Therefore, any $g \in L^2[-\frac{1}{4}, \frac{1}{4})$ has the two representations

$$\begin{aligned} g &= \sum_{n\in\mathbb{Z}} \langle g, e_{2n}\rangle e_{2n} \\ g &= \sum_{n\in\mathbb{Z}} \langle g, e_{2n+1}\rangle e_{2n+1} \end{aligned}$$

Therefore in $L^2[-\frac{1}{4},\frac{1}{4})^2$, for a chosen $e_m$,

$$\begin{aligned} g\otimes e_m &= \sum_{n\in\mathbb{Z}}\langle g,e_{2n}\rangle e_{2n}\otimes e_m \\ g\otimes e_m &= \sum_{n\in\mathbb{Z}}\langle g,e_{2n+1}\rangle e_{2n+1}\otimes e_m \end{aligned}$$

Thus $\{M_{k,n}\eta_0 : k,n\in\mathbb{Z}\}$ is not a Riesz sequence. Furthermore,

$$0=\sum_{n\in\mathbb{Z}}\langle g,e_{2n}\rangle e_{2n}\otimes e_m-\sum_{n\in\mathbb{Z}}\langle g,e_{2n+1}\rangle e_{2n+1}\otimes e_m, \tag{4.4}$$

but the $\ell^2$-norm of the coefficients in (4.4) equals $2\|g\|_{L^2[-\frac{1}{4},\frac{1}{4})}$. Thus we would obtain that for the choice of sequence $\mathbf{c}=\{c_{k,m}:=\langle g,e_k\rangle\delta_m\}$ and any $f$, the mapping

$$\Phi_f:\mathbf{c}\mapsto Hf=0,$$

would not be stable because $\|\mathbf{c}\|=2\|g\|_{L^2[-\frac{1}{4},\frac{1}{4})}\neq 0$.

Nevertheless, the collection of operators $\mathcal{H}=\{H:\operatorname{supp}\eta_H\subset[-\frac{1}{4},\frac{1}{4})^2\}$ is identifiable according to [KP06]. However, we are dealing in reality with a non-trivial representation of the zero operator. Of course, we must observe the fact that there is a canonical ONB for $\mathcal{H}$, with basis elements operators with spreading functions corresponding to the basis elements of the tensor ONB $\{e_{2n}\otimes e_{2m}, m,n\in\mathbb{Z}\}$.

## 4.2 Preliminaries

In this section we consider operators $H$ whose spreading function $\eta_H$belongs to a modulation space $M^1_s(\mathbb{R}^{2d})$. They behave like 'time-frequency localization' operators, in other words, for a distribution $f\in M^\infty(\mathbb{R}^d)$, $Hf$ has a certain decay in the time- and in frequency domains.

First we look at operators whose spreading function is a tensor product.

**Lemma 4.3 ([KP06])** *Let $p,q\in C_c^\infty(\mathbb{R}^d)$. Consider the operator $P$ with spreading function $\eta_P=p\otimes q$. Then there exist functions $\psi_1,\psi_2\in\mathcal{S}(\mathbb{R}^d)$ such that for all $f\in S_0(\mathbb{R}^d)$,*

$$\begin{aligned} |Pf(x)|&\le\|f\|_{S_0}|\psi_1(x)| \\ |\mathcal{F}Pf(\xi)|&\le\|f\|_{S_0}|\psi_2(\xi)| \end{aligned} \tag{4.5}$$

*Proof.* We first estimate the decay of $Pf$ and $\mathcal{F}Pf$

$$
\begin{aligned}
|Pf(x)| &= \left|\iint p(t)q(\nu)f(x-t)e^{2\pi i\nu(x-t)}\,d(t,\nu)\right| \\
&= \left|\int p(t)f(x-t)\left(\int q(\nu)e^{2\pi i\nu(x-t)}d\nu\right)dt\right| \\
&= \left|\int p(t)f(x-t)\widehat{q}(t-x)\,dt\right| \\
&\leq \|f\|_{S_0'}\cdot\|p\cdot T_x\widehat{q}\|_{S_0}. \qquad (4.6)
\end{aligned}
$$

Since $p\cdot T_x\widehat{q}\in\mathcal{S}(\mathbb{R}^d)$, the $S_0$-norm in (4.6) is finite. We need to show that there exists $\psi_1\in\mathcal{S}(\mathbb{R})$ such that $|Pf(x)|\leq\psi_1(x)\|f\|_{S_0'}$. Observe that $\|p\cdot T_x\widehat{q}\|_{S_0}$ tends to 0 at infinity (the support of $p$ is compact, and $\widehat{q}$ has fast decay at $\infty$). We use the alternative definition of $S_0$ as a Wiener amalgam space $W(A(\mathbb{R}^d),\ell^1(\mathbb{Z}^d))$, (2.8) and the equivalent norm. We choose a $\psi\in\mathcal{S}(\mathbb{R}^d)$, such that $\widehat{\psi}\subset[-1,1]^d$ and $\{T_n\widehat{\psi}:n\in\mathbb{Z}^d\}$ form a partition of unity. Then for all $k>1$, there exists a constant $C_k$ such that

$$
\begin{aligned}
\|f\cdot T_x\widehat{\psi}\|_{A(\mathbb{R}^d)} &= \|\mathcal{F}(f\cdot T_x\widehat{\psi})\|_1 = \|\widehat{f}*M_{-x}\psi\|_1 \\
&= \int |V_{\bar{\widehat{\psi}}}f(x,y)|dy \\
&\leq \int C_k(1+|x|+|y|)^{-k}dy \qquad (4.7) \\
&\leq \tilde{C}_k(1+|x|)^{1-k} \qquad (4.8)
\end{aligned}
$$

The estimate in (4.7) follows from [Grö01]: (11.2.5).

Thus we can now estimate directly $\|p\cdot T_x\widehat{q}\|_{S_0}$ using in turn the fact that $\operatorname{supp} p\subset[-\frac{R}{2},\frac{R}{2}]^d$ for some fixed $R$ in (4.9), the fact that $A(\mathbb{R}^d)$ is a Banach algebra in (4.10) and applying maximum estimates in (4.11).

$$
\begin{aligned}
\|p\cdot T_x\widehat{q}\|_{S_0} &= \|T_{-x}p\cdot\widehat{q}\|_{S_0} \\
&= \sum_{n\in\mathbb{Z}}\|T_n\widehat{\psi}\cdot(T_{-x}p\cdot\widehat{q})\|_{A(\mathbb{R}^d)} \\
&= \sum_{n\in[-x-1-\frac{R}{2},-x+1+\frac{R}{2}]^d}\|T_n\widehat{\psi}\cdot(T_{-x}p\cdot\widehat{q})\|_{A(\mathbb{R}^d)} \qquad (4.9) \\
&\leq \|T_xp\|_{A(\mathbb{R})}\sum_{n\in[-x-1-\frac{R}{2},-x+1+\frac{R}{2}]^d}\|T_n\widehat{\psi}\cdot\widehat{q}\|_{A(\mathbb{R}^d)} \qquad (4.10) \\
&\leq \|p\|_{A(\mathbb{R})}C_k(\lfloor 2+R\rfloor+1)^d\max_{n\in[-x-1-\frac{R}{2},-x+1+\frac{R}{2}]}(1+|n|)^{1-k} \qquad (4.11) \\
&\leq \tilde{C}_k\left(2+\min\left\{\lfloor -x-1-\tfrac{R}{2}\rfloor,\lfloor -x+1+\tfrac{R}{2}\rfloor\right\}\right)^{1-k} \qquad (4.12)
\end{aligned}
$$

Equation (4.12) provides the necessary decay of $\|p\cdot T_x\widehat{q}\|_{S_0}$ for any $k>1$ as $|x|\to\infty$. Thus we can choose $\psi_1\in\mathcal{S}$ which dominates the expression in (4.12), such that

$$|Pf(x)| \leq \|f\|_{S_0'}|\psi_1(x)|$$

It remains to estimate the decay in the Fourier domain of $Pf$:,

$$\begin{aligned}
|\mathcal{F}Pf(\xi)| &= |\widehat{p}(\xi) \cdot (q * \widehat{f})(\xi)| \\
&\leq |\widehat{p}(\xi)| \cdot |\langle \widehat{q}, \overline{T_\xi \widehat{f}} \rangle| \\
&\leq |\widehat{p}(\xi)| \cdot \|q\|_{S_0} \cdot \|T_\xi f\|_{S_0'} \\
&= |\widehat{p}(\xi)| \cdot \|q\|_{S_0} \cdot \|f\|_{S_0'}
\end{aligned}$$

Since $p \in \mathcal{S}$ and the Fourier transform is an isomorphism from the Schwarz space $\mathcal{S}$ to itself [Kat76], then $\widehat{p} \in \mathcal{S}$ and $|\widehat{p}(\xi)|$ decays rapidly at infinity. Hence, $|\mathcal{F}Pf(\xi)|$ is bounded by $\psi_2(\xi)\|f\|_{S_0'}$, where $\psi_2(\xi) = |\widehat{p}(\xi)| \cdot \|q\|_{S_0}$ is a rapidly decaying function. □

Lemma 4.3 and Corollary 2.9 form the basis for the remaining statements in Section 4. We consider separately the cases $d = 1$, $d > 1$. In the following lemma we consider the case $d = 1$.

**Lemma 4.4** *Let $\eta_0$ be a function from $M_v^1(\mathbb{R}^2)$, where $v(z) = (1 + |z|)^{2+\delta}$ is a polynomial weight. Then the Hilbert-Schmidt operator $H_0$ associated to $\eta_0$ satisfies the following*

- *There exists $\varphi_1(x) = O(x^{-2-\delta})$ such that $|H_0 f(x)| \leq \varphi_1(x)\|f\|_{M^\infty}$ for all $f \in M^\infty(\mathbb{R})$.*
- *There exists $\varphi_2(\xi) = O(\xi^{-2-\delta})$ such that $|\mathcal{F}H_0 f(\xi)| \leq \varphi_2(\xi)\|f\|_{M^\infty}$ for all $f \in M^\infty(\mathbb{R})$.*

**Note:** The decay estimates in fact show that $H_0 f \in L^2(\mathbb{R}^2)$, see [Fol99] (2.52).

*Proof.* We consider a spreading function $\eta_P = p \otimes q$ with Hilbert-Schmidt operator $P$. With a careful choice of initial parameters $a, b, c, d$ and functions $p, q \in C_c^\infty(\mathbb{R})$, we are able to obtain a Gabor frame $(\eta_P, a\mathbb{Z} \times b\mathbb{Z} \times c\mathbb{Z} \times d\mathbb{Z})$ for $L^2(\mathbb{R}^2)$. Because $\eta_P \in \mathcal{S}(\mathbb{R}^2) \subset M_v^1(\mathbb{R}^2)$, in fact the Gabor system $(\eta_P, a\mathbb{Z} \times b\mathbb{Z} \times c\mathbb{Z} \times d\mathbb{Z})$ is a universal Banach frame according to the definition of Gröchenig for all modulation spaces $M_v^1(\mathbb{R}^2)$ ([Grö01], Chapter 13.6). Due to the inverse closure of the Banach algebra $(\ell_v^1(\mathbb{Z}^4), \natural)$ for polynomial weights $v$ [GL03], the dual window $\tilde{\eta}_P \in M_v^1(\mathbb{R}^2)$. So the series expansion

$$g = \sum_{k,l,m,n} \langle g, T_{ak,bl} M_{cm,dn} \tilde{\eta}_P \rangle T_{ak,bl} M_{cm,dn} \eta_P, \quad g \in M_v^1(\mathbb{R}^2) \tag{4.13}$$

holds with convergence in the $M_v^1$-norm and the inequality

$$A' \|g\|_{M_v^1} \leq \|\{\langle g, T_{ak,bl} M_{cm,dn} \tilde{\eta}_P \rangle\}\|_{\ell_v^1} \leq B' \|g\|_{M_v^1} \tag{4.14}$$

holds for all $g \in M_v^1(\mathbb{R}^2)$, see Definition 2.6 and [Grö01]: (12.2.6). Furthermore, because the coefficients have decay stronger than $\ell^1$, the convergence of the series holds in $L^1(\mathbb{R}^2)$ and in $L^2(\mathbb{R}^2)$.

Since the operator $H$ has spreading function $\eta_H \in M_v^1(\mathbb{R}^2)$, equations (4.13) and (4.14) show that

$$\eta_H = \sum_{k,l,m,n} c_{k,l,m,n} T_{ak,bl} M_{cm,dn} \eta_P$$

for some $\mathbf{c} \in \ell_v^1$, where $v(z) = (1+|z|)^{2+\delta}$ is a polynomial weight. This shows that $\eta_H \in M_v^1 \subset M^1$. It is legitimate to use as identifier distributions $f \in M^\infty$, because

$$\mathcal{S} \subset M_v^1 \subset M^1 \subset M^\infty \subset M_{\frac{1}{v}}^\infty \subset \mathcal{S}',$$

a consequence of Lemma 11.3.6 and 12.1.10 from [Grö01] (the constant weight 1 is $(1+|z|)^{2+\delta}$-moderate - see Lemma 11.1.1 from [Grö01], which proves the inclusion $M_v^1 \subset M^1$).

Next, we estimate the decay of $Hf$ in the time and frequency domains. We shall use the fact that translation and modulation are isometries on $M^1$ and hence also on $M^\infty$ and make the following estimates using the result from Lemma 4.3.

$$\begin{aligned} |Hf(x)| &= |\sum_{k,l,m,n} c_{k,l,m,n} T_{ak} M_{-cm} T_{-dn} P T_{dn} M_{bl-cm} f(x)| \\ &\leq \sum_{k,l,m,n} |c_{k,l,m,n}| \cdot T_{ak-dn} |P T_{dn} M_{bl-cm} f(x)| \\ &\leq \|f\|_{M^\infty} \sum_{k,l,m,n} |c_{k,l,m,n}| \cdot T_{ak-dn} \psi_1(x) \end{aligned} \tag{4.15}$$

Since $\{|c_{k,l,m,n}|\}_{k,l,m,n} := \mathbf{c} \in \ell_v^1(\mathbb{Z}^4) \subset \ell^1(\mathbb{Z}^4)$, we claim that the expression on the right-hand side of (4.15), which for the sake of clarity we denote

$$\varphi_1(x) = \sum_{k,l,m,n} |c_{k,l,m,n}|\, T_{ak-dn} \psi_1(x) = \sum_{k,n} \tilde{c}_{k,n} T_{ak-dn} \psi_1(x),$$

inherits the decay and is $O(|x|^{-2-\delta})$.

Namely let us make the change of variables $x^{2+\delta} = y^2$, for $x \geq 0, y \in \mathbb{R}$ and $x^{2+\delta} = -y^2$, for $x < 0$. Then

$$\sup_{x>0} |x^{2+\delta} \sum_{k,n} \tilde{c}_{k,n} T_{ak-dn} \psi_1(x)| = \sup_{y>0} |y^2 \sum_{k,n} \tilde{c}_{k,n} T_{ak-dn} \psi_1(y^{\frac{2}{2+\delta}})|$$

Since $y^{\frac{2}{2+\delta}}$ is monotonic on $\mathbb{R}$ and due to our choice $\psi_1 \in \mathcal{S}$ (i.e. $\psi_1$ decays faster than the reciprocal of any polynomial on $\mathbb{R}$), $\tilde{\psi}_1(y) = \psi_1(y^{\frac{2}{2+\delta}})$ also decays faster than the reciprocal of any polynomial. Then

$$\begin{aligned}\sup_y |y^2 \sum_{k,n} \tilde{c}_{k,n} T_{ak-dn} \tilde{\psi}_1(y)| \leq \sup_y | \sum_{k,n} \tilde{c}_{k,n} T_{ak-dn} (\tilde{\psi}_1(y) \cdot y^2)| + \\ + \; 2 \sup_y | \sum_{k,n} \tilde{c}_{k,n} (ak - dn) T_{ak-dn} (\tilde{\psi}_1(y) \cdot y)| \\ + \sup_y | \sum_{k,n} \tilde{c}_{k,n} (ak-dn)^2 T_{ak-dn} \tilde{\psi}_1(y)|\end{aligned} \tag{4.16}$$

We make the following estimate of the bounds of the sum in (4.16)

$$\begin{aligned}\sup_y |y^2 \sum_{k,n} \tilde{c}_{k,n} T_{ak-dn} \tilde{\psi}_1(y)| \leq C \Bigg( \sum_{k,n} |\tilde{c}_{k,n}| \sup_y |(\tilde{\psi}_1(y) \cdot y^2)| \\ + \sum_{k,n} |\tilde{c}_{k,n}| \cdot |ak - dn| \sup_y |(\tilde{\psi}_1(y) \cdot y)| \\ + \sum_{k,n} |\tilde{c}_{k,n}| \cdot |ak - dn|^2 \sup_y |\tilde{\psi}_1(y)| \Bigg)\end{aligned} \tag{4.17}$$

In (4.17) $C$ is some positive constant coming from the estimates from Lemma 4.3. We analyze separately these summands in (4.18). First because $\tilde{\psi}_1 \cdot y^2$, $\tilde{\psi}_1 \cdot y$, $\tilde{\psi}_1$ belong to the Schwarz class, they are bounded and decay faster than the reciprocal of any polynomial. Second the inequalities

$$\begin{aligned}&\sum_{k,n} |\tilde{c}_{k,n}| \leq \|\mathbf{c}\|_{\ell^1_v} < \infty \\ &\sum_{k,n} |\tilde{c}_{k,n}| \cdot |ak - dn|^2 \leq C_1 \sum_{k,n} |\tilde{c}_{k,n}| (1 + a|k| + d|n|)^2 < \|\mathbf{c}\|_{\ell^1_v} < \infty, \\ &\sum_{k,n} |\tilde{c}_{k,n}| \cdot |ak - dn| \leq C_2 \sum_{k,n} |\tilde{c}_{k,n}| (1 + a|k| + d|n|) < \|\mathbf{c}\|_{\ell^1_v} < \infty\end{aligned} \tag{4.18}$$

hold for some constants $C_1, C_2 > 0$ due to the choice $\mathbf{c} \in \ell^1_v$. Thus the whole expression on the right-hand side of (4.17) is bounded, implying the desired decay rate of $Hf$ for $x > 0$. In a similar fashion we prove the decay for $x < 0$. Thus,

$$\sup_x |x^{2+\delta} \phi_1(x)| < C.$$

Then $|Hf(x)| \leq \|f\|_{M^\infty} \varphi_1(x)$ has decay $O(x^{-2-\delta})$. In such a manner, we also estimate the decay in of the Fourier transform of $Hf$

$$\begin{aligned}|\mathcal{F}Hf(\xi)| &= | \sum_{k,l,m,n} c_{k,l,m,n} M_{-ak} T_{-cm} M_{dn} \mathcal{F} P T_{dn} M_{bl-cm} f(\xi)| \\ &\leq \sum_{k,l,m,n} |c_{k,l,m,n}| \cdot T_{-cm} |\mathcal{F} P T_{dn} M_{bl-cm} f(\xi)| \\ &\leq \|f\|_{M^\infty} \sum_{k,l,m,n} |c_{k,l,m,n}| \cdot T_{-cm} \psi_2(\xi)\end{aligned} \tag{4.19}$$

Since $\psi_2 \in \mathcal{S}(\mathbb{R})$, and $\mathbf{c} \in \ell_v^1(\mathbb{Z}^4)$, then the expression on the right-hand side of (4.19), which we denote by

$$\varphi_2(x) := \sum_{k,l,m,n} |c_{k,l,m,n}|\, T_{-cm}\psi_2(x),$$

is proven in a similar fashion to have decay $O(x^{-2-\delta})$. So $|\mathcal{F}Hf(\xi)| \leq \varphi_2(\xi)\|f\|_{M^\infty}$ also has decay $O(|\xi|^{-2-\delta})$. □

Thus we are able to cover the case $\eta_0 \in M_v^1(\mathbb{R}^2)$ for polynomial weight $s$ of degree strictly greater than 2.

For $\eta_0$ defined on a higher dimensional space, the parameters of decay have to be adjusted.

**Lemma 4.5** *Let $\eta_0$ be a function from $M_v^1(\mathbb{R}^{2d})$, where $v_s(z) = (1+|z|)^s, s \geq 2(d+1)$ is a polynomial weight. Then the Hilbert-Schmidt operator $H_0$ associated to $\eta_0$ satisfies the following*

- *There exists $\varphi_1(x) = O(|x|^{-2(d+1)})$ such that $|H_0 f(x)| \leq \varphi_1(x)\|f\|_{M^\infty}$ for all $f \in M^\infty(\mathbb{R}^d)$.*
- *There exists $\varphi_2(\xi) = O(|\xi|^{-2(d+1)})$ such that $|\mathcal{F}H_0 f(\xi)| \leq \varphi_2(\xi)\|f\|_{M^\infty}$ for all $f \in M^\infty(\mathbb{R}^d)$.*

**Note:** The decay estimates in fact show that $H_0 f \in L^2(\mathbb{R}^d)$ [Fol99] (2.52).

*Proof.* The line of proof is similar to that of the previous proposition. We use a tensor Gabor window $\eta_P$ to form an expansion of $\eta_H$.

$$\eta_H = \sum_{k,l,m,n} c_{k,l,m,n} T_{ak,bl} M_{cm,dn} \eta_P$$

for some $\mathbf{c} \in \ell_v^1$, where $v_s(z) = (1+|z|)^{2(d+1)}$ is a polynomial weight.

Next, we estimate the decay of $Hf$ in the time and frequency domains as before, using the result from Lemma 4.3.

$$\begin{aligned} |Hf(x)| &= |\sum_{k,l,m,n} c_{k,l,m,n} T_{ak} M_{-cm} T_{-dn} P T_{dn} M_{bl-cm} f(x)| \\ &\leq \sum_{k,l,m,n} |c_{k,l,m,n}| \cdot T_{ak-dn} |P T_{dn} M_{bl-cm} f(x)| \\ &\leq \|f\|_{M^\infty} \sum_{k,l,m,n} |c_{k,l,m,n}| \cdot T_{ak-dn}\psi_1(x) \end{aligned} \tag{4.20}$$

Up to this point, the proof is identical to that of Lemma 4.4 for $d = 1$. However, the rest of the computation is somewhat different due to the higher dimensionality and different restriction on $s$. We claim that the expression on the right-hand side of (4.20), which for the sake of clarity we denote

$$\varphi_1(x) = \sum_{k,l,m,n} |c_{k,l,m,n}|\, T_{ak-dn}\psi_1(x) = \sum_{\lambda} \tilde{c}_\lambda T_\lambda \psi_1(x),$$

inherits the decay of $\eta_0$ and is $O(|x|^{-(2d+1)})$.

Therefore, our aim is showing that the function

$$\sup_{x\in\mathbb{R}^d} |x|^{2(d+1)}|\varphi_1(x)|$$

is bounded on $\mathbb{R}^{2d}$. Similar to the one given in the proof of Lemma 2.6, [Rie88] we can rewrite this as a polynomial in the coordinates $x_1, \dots x_d$ of $x$ because $|x|^{2(d+1)} = (x_1^2 + \dots x_d^2)^{d+1}$. Then the function

$$|x|^{2(d+1)}|\varphi_1(x)| = (x_1^2 + \dots + x_d^2)^{d+1}|\varphi_1(x)| \tag{4.21}$$

can be bounded by a finite sum of terms of the type $|x^{2(d+1)}\varphi_1(x)|$ (in multi-index notation) - a consequence of the triangle inequality applied to (4.21). Then for fixed $x_1, \dots x_d$ and a fixed $d$-tuple $(i_1, \dots i_d)$ we obtain

$$|x_1^{i_1}x_2^{i_2}\dots x_d^{i_d}\varphi_1(x)| = |\sum_{\lambda\in\Lambda} x_1^{i_1}x_2^{i_2}\dots x_d^{i_d}c_\lambda\psi(x_1-\lambda_1, x_2-\lambda_2, \dots, x_d-\lambda_d)| \tag{4.22}$$

For each $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$, the monomial $x_1^{i_1}x_2^{i_2}\dots x_d^{i_d}$ can be expanded as a polynomial in terms of $\lambda_1^{s_1}\dots\lambda_d^{s^d}(x_1-\lambda_1)^{s_{d+1}}\dots(x_d-\lambda_d)^{s_{2d}}$. Observe that the total power of the monomial is $2(d+1)$. Thus (4.22) is bounded above by

$$\begin{aligned}\sum_{|s|=2d} C_s|\sum_{\lambda\in\Lambda} c_\lambda\lambda_1^{s_1}\dots\lambda_d^{s_d} \\ (x_1-\lambda_1)^{s_{d+1}}\dots(x_d-\lambda_d)^{s_{2d}}\psi(x_1-\lambda_1, x_2-\lambda_2, \dots, x_d-\lambda_d)|\end{aligned} \tag{4.23}$$

where the constants $C_s$ come from the polynomial and are independent of $\lambda$. To illustrate this, consider the next two numerical examples. For example when $d = 2$, for a 'cross-product' the expansion looks like

$$
\begin{aligned}
|x_1^2x_2^2\varphi_1(x_1,x_2)| &= |x_1^2x_2^2\sum_\lambda c_\lambda T_\lambda\psi(x_1,x_2)| \\
&\leq |\sum_\lambda c_\lambda(x_1-\lambda_1)^2(x_2-\lambda_2)^2T_\lambda\psi(x_1,x_2)| \\
&+2|\sum_\lambda \lambda_1c_\lambda(x_1-\lambda_1)^2(x_2-\lambda_2)T_\lambda\psi(x_1,x_2)| \\
&+2|\sum_\lambda \lambda_2c_\lambda(x_1-\lambda_1)(x_2-\lambda_2)^2T_\lambda\psi(x_1,x_2)| \\
&+12|\sum_\lambda \lambda_1\lambda_2c_\lambda(x_1-\lambda_1)(x_2-\lambda_2)T_\lambda\psi(x_1,x_2)| \\
&+|\sum_\lambda \lambda_1^2c_\lambda(x_2-\lambda_2)^2T_\lambda\psi(x_1,x_2)| \\
&+|\sum_\lambda \lambda_2^2c_\lambda(x_1-\lambda_1)^2T_\lambda\psi(x_1,x_2)| \\
&+12|\sum_\lambda \lambda_1^2\lambda_2c_\lambda(x_2-\lambda_2)T_\lambda\psi(x_1,x_2)| \\
&+12|\sum_\lambda \lambda_1\lambda_2^2c_\lambda(x_1-\lambda_1)T_\lambda\psi(x_1,x_2)| \\
&+13|\sum_\lambda \lambda_1^2\lambda_2^2c_\lambda T_\lambda\psi(x_1,x_2)|,
\end{aligned}
\tag{4.24}
$$

while for a term $x^4$ we have

$$
\begin{aligned}
|x^4\varphi_1(x)| &= |x^4\sum_\lambda c_\lambda T_\lambda\psi(x)| \\
&\leq |\sum_\lambda c_\lambda(x-\lambda)^4T_\lambda\psi(x)| + 4|\sum_\lambda \lambda c_\lambda(x-\lambda)^3T_\lambda\psi(x)| \\
&+6|\sum_\lambda \lambda^2c_\lambda(x-\lambda)^2T_\lambda\psi(x)| + 4|\sum_\lambda \lambda^3c_\lambda(x-\lambda)T_\lambda\psi(x)| \\
&+|\sum_\lambda \lambda^4c_\lambda T_\lambda\psi(x)|
\end{aligned}
\tag{4.25}
$$

For each term of (4.23) we can apply similar estimates as those in (4.16) and (4.17). That is,

$$
\begin{aligned}
&\sup_x|\sum_{\lambda\in\Lambda}c_\lambda\lambda_1^{s_1}\dots\lambda_d^{s_d}(x_1-\lambda_1)^{s_{d+1}}\dots(x_d-\lambda_d)^{s_{2d}}\times \\
&\qquad\psi(x_1-\lambda_1,x_2-\lambda_2,\dots,x_d-\lambda_d)| \\
&\quad\leq\sum_{\lambda\in\Lambda}|c_\lambda\lambda_1^{s_1}\dots\lambda_d^{s_d}|\sup_x|(x_1-\lambda_1)^{s_{d+1}}\dots(x_d-\lambda_d)^{s_{2d}}\times \\
&\qquad\psi(x_1-\lambda_1,x_2-\lambda_2,\dots,x_d-\lambda_d)|
\end{aligned}
\tag{4.26}
$$

Then, $\sum_{\lambda\in\Lambda} |c_\lambda \lambda_1^{s_1} \dots \lambda_d^{s_d}|$ is convergent because $c_\lambda \in \ell_s^1(\mathbb{Z}^{2d})$ (as in (4.16) and (4.17), the decay of $(c_\lambda)$ absorbs all terms in $\lambda^{2(d+1)}$ (in multi-index notation)). On the other hand,

$$\sup_x |(x_1-\lambda_1)^{s_{d+1}} \dots (x_d-\lambda_d)^{s_{2d}} \psi(x_1-\lambda_1, x_2-\lambda_2, \dots, x_d-\lambda_d)|$$

is bounded on $\mathbb{R}^d$ because $\psi \in \mathcal{S}(\mathbb{R}^d)$. Therefore, we obtain boundedness of each summand in (4.23), which implies that $\sup_{x\in\mathbb{R}^d} |x|^{2(d+1)}|\varphi_1(x)|$ is bounded on $\mathbb{R}^{2d}$. Therefore, $\varphi_1(x) = O(|x|^{-2(d+1)})$, and the boundedness estimate for $Hf(x)$ is proven.

Similarly we show the boundedness estimate for $\mathcal{F}Hf(\xi)$. □

The rates of decay of $Hf$ and $\mathcal{F}Hf$ obtained in Lemmas 4.4 and Lemma 4.5 together with Lemma 4.7 are necessary for applications in Section 4.4 and 4.5.

## 4.3 Two matrix lemmas

In this section we list two lemmas about non-existence of a left-inverse of bi-infinite matrices.

**Lemma 4.6** *Let $M = (m_{j,k}) : \ell^2(\mathbb{Z}^{2d}) \mapsto \ell^2(\mathbb{Z}^{2d})$ be a bi-infinite matrix, whose action on (a subset of) vectors in $\ell^2(\mathbb{Z}^{2d})$ is bounded. If there exists a monotonically decreasing function $w$ such that $w(x) = O(|x|^{-2d-\delta}), \delta > 0$ such that $|m_{j,k}| \le w(|j|)$, then $M$ does not have a bounded left inverse.*

**Note:** This matrix need not represent a compact operator.

*Proof.* The assumption essentially states that there exists a constant $C$ such that the entries of the $j$-th matrix row $m_j$ are bounded uniformly by $C|j|^{-2d-\delta}$. Therefore we fix $\epsilon > 0$, and choose $K \in \mathbb{N}$ (depending on $\epsilon$) such that

$$\sum_{\|j\|_\infty > K} \max_{\|k\|_\infty \le K+1} |m_{j,k}|^2 \le \int_{|y|\ge K+1} Cy^{-2(2+\delta)} dy = O(K^{-2d}) < \frac{\epsilon^2}{(2K+3)^d},$$

where we have used polar coordinates in the integral:

$$\int_{|y|\ge K+1} Cy^{-2(2d+\delta)} dy = \iint R^{-4d-2\delta} R^{d-1} \sin\theta d(R,\theta) = O(K^{-2d}).$$

($R^{d-1}\sin\theta$ is a shorthand notation for the generalized Jacobian of the coordinate change). Let us take a vector $x \in \ell^2(\mathbb{Z}^2)$ such that $\|x\|_2 = 1$, $x_k = 0$ for $\|k\|_\infty > K+1$ and $x \perp m_j$, $\|j\|_\infty \le K$, where $m_j$ is the $j$-th matrix row. Such a vector always exists because the submatrix $M' = (m_{j,k})_{|j|\le K, |k|\le K+1}$ of $M$ has $(2K+1)^d$ rows which cannot span a $(2K+3)^d$-dimensional vector space. We estimate the $\ell^2$-norm of $Mx$.

$$\|Mx\|_2^2 = \sum_{\|j\|_\infty > K} |\langle x, m_j\rangle|^2 \le \sum_{\|j\|_\infty > K} \|x\|_2^2 \cdot \|R_{K+1} m_j\|_2^2, \tag{4.27}$$

where $R_{K+1}m_j$ is the restriction of the $j$-th row $m_j$ to the elements $m_{j,k} : \|k\|_\infty \leq K+1$. Hence,

$$\|Mx\|_2^2 \leq \sum_{\|j\|_\infty > K} \|R_{K+1}m_j\|_2^2 \leq (2K+3) \sum_{\|j\|_\infty > K+1} \max_{\|k\|_\infty \leq K} |m_{j,k}|^2 \leq \epsilon^2$$

Therefore, for $x$ with $\|x\|_2 = 1$, $\|Mx\|_2 \to 0$ and $M$ does not have a bounded left inverse. □

The second lemma about invertibility of 'skew-diagonal' matrices is a subcase of Theorem 2.1 [Pfa08], see also Lemma 3.5 [KP06].

**Lemma 4.7 ([Pfa08])** *Given $M = (m_{j,k}) : \ell^2(\mathbb{Z}^{2d}) \mapsto \ell^2(\mathbb{Z}^{2d})$. If there exists a monotonically decreasing function $w : \mathbb{R}_0^+ \mapsto \mathbb{R}_0^+$ with $w(x) = O(x^{-2d-\delta}), \delta > 0$ and constants $\lambda > 1$ and $K_0 > 0$ with $|m_{j,k}| < w(\|k - \lambda j\|_\infty)$ for $\|k - \lambda j\|_\infty > K_0$, then $M$ does not have a bounded left inverse.*

The proof of the lemma can be found in [Pfa08] or [KP06].

In the following, we will consider different combinations of $\Lambda$ in Figures 2 and 3 and spreading functions $\eta_0$ belonging to the modulation space $M_v^1(\mathbb{R}^2)$, for a polynomial weight $v(z) = (1 + |z|)^s, s > 2$.

In particular, we explore different combinations of 'initial conditions' for operator families $\mathcal{H}_\Lambda$, where

$$\mathcal{H}_\Lambda = \{H : \eta_H \in \overline{\text{span}}\,(\eta_0, \Lambda)\}.$$

The choice of $\eta_0, \Lambda$ goes beyond the particular example considered in [KP06]. The basis of the analysis of identification of operator families includes the following constraints

(I) The structure of index set $\Lambda$ is important for identification of $\mathcal{H}_\Lambda$ (the measure used is 2-density of $\Lambda$, as defined in Section 4)

(II) The properties of $\eta_0$ and its associated prototype operator $H_0$, especially when $\eta_0 \in M_s^1(\mathbb{R}^{2d})$ for certain $s$ as presented in Section 4 also play an important role.

(III) The requirement that $\{\pi(\lambda)\eta_0 : \lambda \in \Lambda\}$ is a Riesz basis sequence inside $L^2(\mathbb{R}^{2d})$ assures well-posedness of the discretized identification problem.

The study [KP06] considers essentially $\Lambda = 0 \times \alpha\mathbb{Z} \times \beta\mathbb{Z}, \eta_0$ - characteristic function of a fundamental domain of $\Lambda$. In the language of that study $\mathcal{H}_\Lambda$ is denoted operators with spreading symbol of area $ab$.

We want to test whether any of the following statements hold.

(IV) There exists a constant $C > 0$ such that $D_2(\Lambda) > C \Longrightarrow \mathcal{H}_\Lambda$ is not identifiable.

(V) There exists a constants $c > 0$ such that $D_2(\Lambda) < c \Longrightarrow \mathcal{H}_\Lambda$ is identifiable.

In [KP06] both statements are confirmed with constants $C = c = 1$.

In the following we illustrate some typical cases from Tables 2 and 3. The primary goal is to formulate results on a relation between identifiability and non-identifiability and the measure of the *2-density* of $\Lambda$.

## 4.4 Operator families which are never identifiable

Here we list the examples of operators families from Tables 2 and 3, which are not identifiable regardless of any density measure. Recall that $\Phi_f$ denotes the evaluation operator

$$\Phi_f : \mathcal{H} \to L^2(\mathbb{R}), \quad \Phi_f(H) = Hf$$

**Proposition 4.8** *Let $\eta_0 \in M^1_v(\mathbb{R}^2)$, where $v_s(z) = (1 + |z|)^s, s > 2$. The operator family $\mathcal{H}_i = \{H : \eta_H \in \mathcal{J}_i \cap M^1(\mathbb{R}^2)\}$, where*

1. $\mathcal{J}_1 = \overline{\text{span}}\, \{T_{\alpha k,\gamma m}\eta_0 : k, m \in \mathbb{Z}\}$
2. $\mathcal{J}_2 = \overline{\text{span}}\, \{T_{0,\gamma m}M_{0,\beta l}\eta_0 : l, m \in \mathbb{Z}\}$
3. $\mathcal{J}_3 = \overline{\text{span}}\, \{T_{0,\gamma m}M_{\delta n,0}\eta_0 : m, n \in \mathbb{Z}\}$
4. $\mathcal{J}_4 = \overline{\text{span}}\, \{T_{\alpha k,\beta m}M_{0,\beta k} : k, m \in \mathbb{Z}\}$

*is not identifiable.*

*Proof.* The operators in these classes have the following series expansions with respect to the prototype operator $H_0$ with spreading function $\eta_0$:

1. $H = \sum_{k,m\in\mathbb{Z}^d} c_{k,m} T_{\alpha k} H_0 M_{\gamma m}$.
2. $H = \sum_{l,m\in\mathbb{Z}^d} c_{l,m} M_{\beta l} H_0 M_{\gamma m-\beta l}$
3. $H = \sum_{m,n\in\mathbb{Z}^d} c_{m,n} T_{-\delta n} H_0 T_{\delta n} M_{\gamma m}$
4. $H = \sum_{k,m\in\mathbb{Z}^d} c_{k,m} T_{\alpha k} M_{\beta k} H_0 M_{\beta(m-k)}$

To demonstrate non-identifiability of $\mathcal{H}_i$, we consider a subfamily of $\mathcal{H}_i$ and show that it is not identifiable. This implies that the whole family is not identifiable.

Let us give a sketch of logic behind the proof. As we have seen before, the operator $H_0$ acts as a time-frequency localization operator, hence the information carried by the 2-parameter coefficient sequence must be preserved in the respective Gabor expression. However, there is only one parameter in the time-frequency shift outside the action of $H_0$, which means that part of the information disappears ('is erased') under the action of $H_0$ and is unrecoverable.

Fix a natural number $N > 0$. Let $\ell_N = \{(c_i) \in \ell_0(\mathbb{Z}) : c_i = 0, i > N\}$. For the first family $\mathcal{H}_1$ we consider only those operators with coefficients in the sequences $c_{k,m} = \delta_0(k)c_m, (c_m) \in \ell_N$. For the second family $\mathcal{H}_2$ - only those operators with coefficients in the set of sequences $c_{l,m} = \delta_0(l)c_m, (c_m) \in \ell_N$, for $\mathcal{H}_3$ - in $c_{m,n} = \delta_0(n)c_m, (c_m) \in \ell_N$, and for $\mathcal{H}_4$ - $c_{k,m} = \delta_0(k)c_m, (c_m) \in \ell_N$. In fact in all of these cases we obtain a subfamily of operators

$$\mathcal{H}' = \{H = \sum_{|m| \leq N} c_m H_0 M_{\gamma m} : (c_m) \in \ell_N\} \subset \mathcal{H}.$$

We claim that $\mathcal{H}'$ is not identifiable. For this purpose we construct a bounded and invertible analysis operator $C : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2)$ and a bounded and invertible synthesis operator $D : \ell^2(\mathbb{Z}) \to \mathcal{H}$ such that the composition of maps

$$C \circ \Phi_f \circ D : \ell_N(\mathbb{Z}) \to \ell^2(\mathbb{Z}^2), \quad f \in S_0'(\mathbb{R})$$

is not stable for any $f \in S_0'(\mathbb{R})$. The stability of $C, D$ would then imply that $\Phi_f$ is not stable for any $f \in S_0'(\mathbb{R})$. The operator $D$ will be defined as follows

$$D : \ell_N(\mathbb{Z}) \to \mathcal{H}', \quad D : \{c_m\} \mapsto \sum_m c_m H_0 M_{\gamma m}$$

where $H_0$ is the prototype operator with spreading function $\eta_0$. Since the collection of function $\{T_{\alpha k, \gamma m}\eta_0 : k, m \in \mathbb{Z}\}$ is a Riesz sequence in $L^2(\mathbb{R})$, the associated collection of operators $\{H_0 M_{\gamma m} : m \in \mathbb{Z}\}$ (which is a subsequence of a Riesz sequence) forms a Riesz sequence in the space of operators $\mathcal{H}'$. Hence $D$ is well-defined, bounded and invertible on $\ell_0(\mathbb{Z})$, so by density it can be extended to all of $\ell^2(\mathbb{Z})$.

To define a bounded and invertible analysis operator $C : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2)$, we use the normalized Gaussian $\gamma_1$ and the fact that for $1 < (ab)^{-1}$ the Gabor system $(\gamma_1, a\mathbb{Z} \times b\mathbb{Z})$ is a frame for $L^2(\mathbb{R})$ [Lyu92]. Hence the analysis map

$$C : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2) : \quad C : f \mapsto \{\langle f, T_{ak'} M_{bl'} \gamma_1 \rangle\}_{k',l'}$$

is well-defined, bounded and invertible.

The action of the evaluation operator $\Phi_f$ is bounded on the subspace of operators $\mathcal{H}'$, because

$$\begin{aligned}
\|\Phi_f(H)\|_2^2 &= \|\sum_{|m|\le N} c_m H_0 M_{\gamma m} f\|_2^2 \\
&= \int_{\mathbb{R}} |\sum_{|m|\le N} c_m H_0 M_{\gamma m} f(t)|^2 dt \\
&\le \int_{\mathbb{R}} (\sum_{|m|\le N} |c_m| \cdot |H_0 M_{\gamma m} f(t)|)^2 dt \\
&\le \int_{\mathbb{R}} (\sum_{|m|\le N} |c_m| \cdot |\varphi_1(t)| \cdot \|f\|_{S_0'})^2 dt \\
&= \|f\|_{S_0'}^2 \cdot (\sum_{|m|\le N} |c_m|)^2 \cdot \int |\varphi_1(t)|^2 dt \\
&\asymp C(N)\|\mathbf{c}\|_{\ell^2}^2 \\
&\asymp C(N)\|\eta_H\|_2^2 = C(N)\|H\|_{HS}^2,
\end{aligned} \tag{4.28}$$

where for the sake of shortness we denote $\mathbf{c} = \{c_m\}$. Therefore, the pseudo-inverse of $\Phi_f$ is bounded on $\mathcal{H}'$.

The composition of mappings $C \circ \Phi_f \circ D$

$$\{c_m\} \mapsto D(c_m) \mapsto D(c_m)f \mapsto \{\langle D(c_m)f, T_{ak'} M_{bl'} \gamma_1\rangle\} \tag{4.29}$$

can be represented as a matrix acting on the sequence $(c_m)$, More precisely, a bi-infinite matrix $M = (m_{k',l';m})$, where

$$m_{k',l';m} = \langle H_0 M_{\gamma m} f, T_{ak'} M_{bl'} \gamma_1\rangle \tag{4.30}$$

Having represented $C \circ \Phi_f \circ D$ as a matrix we use Lemma 4.6 to show that $M$ is not invertible on $\ell_N$. In other words the matrix action is,

$$(C \circ \Phi_f \circ D(c_m))_{k',l'} = \sum_m c_m m_{k',l';m}$$

We estimate the coefficients of $M$, applying the results from Lemma 4.4 in the following computations.

$$\begin{aligned}
|m_{k',l';m}| &= |\langle H_0 M_{\gamma m} f, T_{ak'} M_{bl'} \gamma_1\rangle| \\
&\le \langle |H_0 M_{\gamma m} f|, T_{ak'} |\gamma_1|\rangle \\
&\le \varphi_1 * \gamma_1(ak') \cdot \|f\|_{S_0'}
\end{aligned} \tag{4.31}$$

In a similar manner we can obtain an alternative estimate, by taking Fourier transform on both sides of the inner product (4.30), and apply the estimate from

Lemma 4.4.

$$
\begin{aligned}
|m_{k',l';m}| &= |\langle \mathcal{F}H_0 M_{\gamma m} f, M_{ak'} T_{-bl'} \gamma_1 \rangle| \\
&\leq \langle |\mathcal{F}H_0 M_{\gamma m} f|, T_{-bl'} |\gamma_1| \rangle \\
&\leq \varphi_2 * \gamma_1(bl') \cdot \|f\|_{S_0'}
\end{aligned}
\tag{4.32}
$$

Since $\varphi_1, \varphi_2$ are positive and decay faster than $x^{-2}$, so do the respective convolutions $\varphi_1 * \gamma_1$ and $\varphi_2 * \gamma_1$. So we can define

$$h(x) = \max(\varphi_1 * \gamma_1(ax), \varphi_1 * \gamma_1(-ax), \varphi_2 * \gamma_1(bx), \varphi_2 * \gamma_1(-bx) \cdot \|f\|_{S_0'}).$$

Then it is clear that

$$|m_{k',l';m}| \leq h(\max\{|k'|, |l'|\}) = h(\|\mathbf{z}\|_\infty),$$

where we label $\mathbf{z} = (k', l')$.

A straightforward application of Lemma 4.6 shows that the matrix $M = (m_{\mathbf{z};m})$ with $|m_{\mathbf{z};m}| = O(\mathbf{z}^{-2-\delta})$ does not have a bounded left inverse on the subset $\{\mathbf{c} : c_m = 0, |m| > N\}$. This brings us to a contradiction. Therefore, we can conclude that under the conditions of Proposition 4.8, the operator classes $\mathcal{H}_i$ are not identifiable. □

Next we look at other examples of operator families.

**Proposition 4.9** *Let $\eta_0 \in M_v^1(\mathbb{R}^2)$, where $v_s(z) = (1 + |z|)^s, s > 2$. The operator class $\mathcal{H}_i = \{H : \eta_H \in \mathcal{J}_i \cap M^1(\mathbb{R}^2)\}$, where*

1. $\mathcal{J}_1 = \overline{\text{span}}\, \{T_{\alpha k, \beta m} M_{\alpha k, 0} \eta_0 : k, m \in \mathbb{Z}\}$
2. $\mathcal{J}_2 = \overline{\text{span}}\, \{T_{\alpha k, \beta l} M_{\alpha k, \beta l} : k, l \in \mathbb{Z}\}$
3. $\mathcal{J}_3 = \overline{\text{span}}\, \{T_{\alpha k, \beta k} M_{\alpha n, 0} \eta_0 : k, n \in \mathbb{Z}\}$
4. $\mathcal{J}_4 = \overline{\text{span}}\, \{T_{\alpha k, \beta n} M_{\alpha n, 0} \eta_0 : k, n \in \mathbb{Z}\}$

*is not identifiable.*

*Proof.* The operators in these classes have the following series expansions with respect to the prototype operator $H_0$ with spreading function $\eta_0$:

1. $H = \sum_{k,m \in \mathbb{Z}^d} c_{k,m} H_0 T_{\alpha k} M_{\beta m}$
2. $H = \sum_{k,l \in \mathbb{Z}^d} c_{k,l} M_{\beta l} H_0 T_{\alpha k}$
3. $H = \sum_{k,n \in \mathbb{Z}^d} c_{k,n} T_{\alpha(k-n)} H_0 T_{\alpha n} M_{\beta k}$
4. $H = \sum_{k,n \in \mathbb{Z}^d} c_{k,n} T_{\alpha(k-n)} H_0 T_{\alpha n} M_{\beta n}$

The idea of the proof again is to consider a subfamily of $\mathcal{H}_i$ and show that the subfamily is not identifiable. Then it will follow that the whole family $\mathcal{H}_i$ is not identifiable. This was used in Proposition 4.8.

Fix a natural number $N > 0$. For the first family $\mathcal{H}_1$ we consider only those operators with coefficients in the sequences $\mathbf{c} \in \ell^2(\mathbb{Z}) : c_{k,m} = \delta_0(m)c_k : (c_k) \in \ell_N(\mathbb{Z})$, for $\mathcal{H}_2$ - in $\mathbf{c} \in \ell^2(\mathbb{Z}) : c_{k,l} = \delta_0(l)c_k : (c_k) \in \ell_N(\mathbb{Z})$. For the $\mathcal{H}_3, \mathcal{H}_4$ we pick only those operators with coefficients in the set of sequences $c_{k,n} = \delta_0(k-n)c_k \in \ell^2(\mathbb{Z}) : (c_k) \in \ell_N(\mathbb{Z})$. The resulting operators form the subfamilies

$$\mathcal{H}'_{1,2} = \{H = \sum_{k\in\mathbb{Z}} c_k H_0 T_{\alpha k} M_{\beta k} : (c_k) \in \ell_N(\mathbb{Z})\} \subset \mathcal{H},$$

in the first and second cases and

$$\mathcal{H}'_{3,4} = \{H = \sum_{k\in\mathbb{Z}} c_k H_0 T_{\alpha k} : (c_k) \in \ell_N(\mathbb{Z})\} \subset \mathcal{H},$$

in the third and fourth. From here the proof follows the steps of the proof of Proposition 4.8.

□

## 4.5 Identifiability depends on density

In the following cases we show that 2-density plays a role in determining whether operator families are identifiable or not.

**Proposition 4.10** *Let $\eta_0 \in M^1_v(\mathbb{R}^2)$, where $v_s(z) = (1+|z|)^s, s > 2$. The operator class $\mathcal{H}_i = \{H : \eta_H \in \mathcal{J}_i \cap M^1(\mathbb{R}^2)\}$ where*

1. *$\mathcal{J}_1 = \overline{\operatorname{span}}\{M_{\alpha k,\beta l}\eta_0 : k,l \in \mathbb{Z}\}$, arising from the index set $\Lambda_1 = (0,0,\alpha k,\beta l) : k,l \in \mathbb{Z}$.*

2. *$\mathcal{J}_2 = \overline{\operatorname{span}}\{T_{\alpha k,0}M_{0,\beta l} : k,l \in \mathbb{Z}\}$, arising from the index set $\Lambda_2 = (\alpha k,0,0,\beta l) : k,l \in \mathbb{Z}$*

*is not identifiable if the 2-density of $\Lambda_i$ is greater than 1.*

These families are listed as B2, B6 in Table 2.

*Proof.* The condition on the 2-density for families $\mathcal{H}_1$ and $\mathcal{H}_2$

$$D_2(\Lambda_i) = \frac{1}{|\alpha\beta|} > 1$$

implies that $|\alpha\beta| < 1$. We shall consider only family $\mathcal{H}_1$ as the line of proof for $\mathcal{H}_2$ is analogous. Without loss of generality we may assume $\alpha, \beta > 0$.

We construct a bounded and invertible analysis operator $C : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2)$ and a bounded and invertible synthesis operator $D : \ell^2(\mathbb{Z}^2) \to \mathcal{H}$ such that the composition

$$C \circ \Phi_f \circ D : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2), \quad f \in S_0'(\mathbb{R})$$

is not stable for any $f \in S_0'(\mathbb{R})$. The stability of $C, D$ would then imply that $\Phi_f$ is not stable for any $f \in S_0'(\mathbb{R})$.

The operator $D$ will be defined as follows

$$D : \ell_0(\mathbb{Z}^2) \to \mathcal{H}, \quad D : \{c_{k,l}\} \mapsto \sum_{k,l} c_{k,l} M_{\alpha k} T_{\beta l} H_0 T_{-\beta l} M_{-\alpha k}$$

where $H_0$ is the prototype operator with spreading function $\eta_0$. Since the collection of function $\{M_{\alpha k, \beta l}\eta_0 : k, l \in \mathbb{Z}\}$ is a Riesz sequence in $L^2(\mathbb{R})$, the associated collection of operators $\{M_{\alpha k} T_{\beta l} H_0 T_{-\beta l} M_{-\alpha k} : k, l \in \mathbb{Z}\}$ forms a Riesz sequence in the space of operators $\mathcal{H}$. Hence $D$ is well-defined, bounded and invertible on $\ell_0(\mathbb{Z}^2)$, so by density it can be extended to all of $\ell^2(\mathbb{Z}^2)$.

To define a bounded and invertible analysis operator $C : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^2)$, we use the normalized Gaussian $\gamma_1$ and the fact that we can choose some $\lambda$ such that $1 < \lambda^2 < (\alpha\beta)^{-1}$ so that $\lambda^2 \alpha\beta < 1$. Then we know that the Gabor system $(\gamma_1, \lambda\beta\mathbb{Z} \times \lambda\alpha\mathbb{Z})$ is a frame. Hence the analysis map

$$C : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2) : \quad C : f \mapsto \{\langle f, M_{\lambda\alpha k'} T_{\lambda\beta l'} \gamma_1 \rangle\}_{k',l'}$$

is well-defined, bounded and invertible.

The combined result of these mappings

$$\{c_{k,l}\} \mapsto D(c_{k,l}) \mapsto D(c_{k,l})f \mapsto \{\langle D(c_{k,l})f, M_{\lambda\alpha k'} T_{\lambda\beta l'} \gamma_1 \rangle\} \tag{4.33}$$

is in fact the result of the action of a bi-infinite matrix $M = (m_{k',l';k,l})$, where

$$m_{k',l';k,l} = \langle M_{\alpha k} T_{\beta l} H_0 T_{-\beta l} M_{-\alpha k} f, M_{\lambda\alpha k'} T_{\lambda\beta l'} \gamma_1 \rangle \tag{4.34}$$

on the sequence $\{c_{k,l}\}$. In other words,

$$(C \circ \Phi_f \circ D(c_{k,l}))_{k',l'} = \sum_{k,l} c_{k,l} m_{k',l';k,l}$$

This is a matrix representation of the map $C \circ \Phi_f \circ D$. We show that the matrix coefficients of $M$ satisfy the requirements of Lemma 4.7. We estimate the coefficients of $M$, applying the bound on $|H_0 T_{-\beta l} M_{-\alpha k} f|$ from Lemma 4.4 in the following computations.

$$\begin{aligned} |m_{k',l';k,l}| &= |\langle M_{\alpha k} T_{\beta l} H_0 T_{-\beta l} M_{-\alpha k} f, M_{\lambda\alpha k'} T_{\lambda\beta l'} \gamma_1 \rangle| \\ &\leq \langle T_{\beta l} |H_0 T_{-\beta l} M_{-\alpha k} f|, T_{\lambda\beta l'} |\gamma_1| \rangle \\ &= |H_0 T_{-\beta l} M_{-\alpha k} f| * \gamma_1(\beta(l - \lambda l')) \\ &\leq \varphi_1 * \gamma_1(\beta(l - \lambda l')) \cdot \|f\|_{S_0'} \end{aligned} \tag{4.35}$$

To obtain a bound involving $k, k'$, we take the Fourier transform on both sides of the inner product in (4.34) and apply the estimate for $|\mathcal{F}H_0T_{-\beta l}M_{-\alpha k}f|$ from Lemma 4.4.

$$\begin{aligned} |m_{k',l',k,l}| &= |\langle T_{\alpha k}M_{-\beta l}(\mathcal{F}H_0T_{-\beta l}M_{-\alpha k}f), T_{\lambda\alpha k'}M_{-\lambda\beta l'}\gamma_1\rangle| \\ &\leq \langle T_{\alpha k}|\mathcal{F}H_0T_{-\beta l}M_{-\alpha k}f|, T_{\lambda\alpha k'}|\gamma_1|\rangle \\ &= |\mathcal{F}H_0T_{-\beta l}M_{-\alpha k}f| * \gamma_1(\alpha(k-\lambda k')) \\ &\leq \varphi_2 * \gamma_1(\alpha(k-\lambda k'))\cdot \|f\|_{S_0'} \end{aligned} \tag{4.36}$$

Since $\varphi_1, \varphi_2$ are positive and have decay greater than $|x|^{-s}$, $s > 2$, so do the respective convolutions $\varphi_1 * \gamma_1$ and $\varphi_2 * \gamma_1$. So we can define

$$h(x) = \max\{\varphi_1 * \gamma_1(\beta x), \varphi_1 * \gamma_1(-\beta x), \varphi_2 * \gamma_1(\alpha x), \varphi_2 * \gamma_1(-\alpha x)\} \cdot \|f\|_{S_0'}.$$

Then it is clear that

$$|m_{k',l',k,l}| \leq h(\max\{|k-\lambda k'|, |l-\lambda l'|\}),$$

so we are in a position to apply Lemma 4.7 and conclude that $M$ is not left-invertible. This brings us to a contradiction. □

**Proposition 4.11** *Let $\eta_0 \in M^1_v(\mathbb{R}^2)$, where $v_s(z) = (1+|z|)^s, s > 2$. The operator class $\mathcal{H}_i = \{H : \eta_H \in \mathcal{J}_i \cap M^1(\mathbb{R}^2)\}$ where*

1. *$\mathcal{J}_1 = \overline{\text{span}}\{T_{\alpha k,\beta l}M_{0,\beta l}\eta_0 : k, l \in \mathbb{Z}\}$, arising from the index set $\Lambda_1 = (\alpha k, \beta l, 0, \beta l) : k, l \in \mathbb{Z}$.*

2. *$\mathcal{J}_2 = \overline{\text{span}}\{T_{\alpha k,\beta m}M_{\alpha m,\beta k} : k, m \in \mathbb{Z}\}$, arising from the index set $\Lambda_2 = (\alpha k, \beta m, \alpha m, \beta k) : k, m \in \mathbb{Z}$.*

3. *$\mathcal{J}_3 = \overline{\text{span}}\{T_{\alpha k,\beta k}M_{\alpha l,\beta l} : k, l \in \mathbb{Z}\}$, arising from the index set $\Lambda_3 = (\alpha k, \beta k, \alpha l, \beta l) : k, l \in \mathbb{Z}$.*

4. *$\mathcal{J}_4 = \overline{\text{span}}\{T_{0,\beta l}M_{\alpha n,\beta l} : l, n \in \mathbb{Z}\}$, arising from the index set $\Lambda_4 = (0, \beta l, \alpha n, \beta l) : l, n \in \mathbb{Z}$.*

*is not identifiable if the 2-density of $\Lambda_i$ is greater than $\frac{1}{\sqrt{2}}$.*

These families are listed as E1, F2, F3, G1 in Table 3.

*Proof.* The condition on the 2-density for families $\mathcal{H}_i : i = 1, 2, 3, 4$

$$D_2(\Lambda_i) = \frac{1}{\sqrt{2}|\alpha\beta|} > \frac{1}{\sqrt{2}}$$

implies that $|\alpha\beta| < 1$. The proof is analogous to that of the Proposition 4.10. □

**Proposition 4.12** *Let $\eta_0 \in M^1_v(\mathbb{R}^2)$, where $v_s(z) = (1+|z|)^s, s > 2$. The operator class $\mathcal{H}_i = \{H : \eta_H \in \mathcal{J}_i \cap M^1(\mathbb{R}^2)\}$ where*

1. $\mathcal{J}_1 = \overline{\operatorname{span}}\,\{T_{\alpha k,\beta k}M_{0,\beta l} : k,l \in \mathbb{Z}\}$, *arising from the index set* $\Lambda_1 = (\alpha k, \beta k, 0, \beta l) : k,l \in \mathbb{Z}$.

2. $\mathcal{J}_2 = \overline{\operatorname{span}}\,\{T_{0,\beta m}M_{\alpha m,\beta l} : l,m \in \mathbb{Z}\}$, *arising from the index set* $\Lambda_2 = (0, \beta m, \alpha m, \beta l) : l,m \in \mathbb{Z}$.

3. $\mathcal{J}_3 = \overline{\operatorname{span}}\,\{T_{\alpha k,0}M_{\alpha l,\beta l} : k,l \in \mathbb{Z}\}$, *arising from the index set* $\Lambda_3 = (\alpha k, 0, \alpha l, \beta l) : k,l \in \mathbb{Z}$.

4. $\mathcal{J}_4 = \overline{\operatorname{span}}\,\{T_{\alpha k,0}M_{\alpha n,\beta k} : k,n \in \mathbb{Z}\}$, *arising from the index set* $\Lambda_4 = (\alpha k, 0, \alpha n, \beta k) : k,n \in \mathbb{Z}$.

*is not identifiable if* $|\alpha\beta| < 1$.

These are families E2, G3, H1, H2 from Table 3.

*Proof.* Essentially analogous to that of Propositions 4.10 and 4.11. However, here we note that the condition $|\alpha\beta| < 1$ cannot be expressed in terms of 2-density of the index set $\Lambda_i$, which is $D_2(\Lambda_i) = \frac{1}{|\alpha|\sqrt{\alpha^2+\beta^2}}$ for $i = 3, 4$ and $D_2(\Lambda_i) = \frac{1}{|\beta|\sqrt{\alpha^2+\beta^2}}$ for $i = 1, 2$. In fact, for any $\epsilon > 0$, we can find $\alpha, \beta$ with $|\alpha\beta| < 1$ such that $D_2(\Lambda_i) < \epsilon$ in the respective cases. For instance, choose $\alpha = 10^{10}, \beta = (10^{10}+1)^{-1}$. Then $|\alpha\beta| < 1$, so the family $\mathcal{H}_1$ is not identifiable by Proposition 4.12, but the 2-density of its index set $D_2(\Lambda_1) \approx 10^{-20}$ is very small. □

Next we state a generalization of case B2 where the sampling set $\Lambda = 0 \times A\mathbb{Z}^{2d}$. We consider sampling points in the modulation domain indexed by a general lattice in $\mathbb{R}^{2d}$ defined by a matrix $A \in \mathrm{GL}(\mathbb{R}, 2d)$.

**Proposition 4.13** *Let* $\eta_0 \in M^1_v(\mathbb{R}^{2d})$, *where* $v_s(z) = (1+|z|)^s, s > 2\,(d=1)$, *and* $s \geq 2(d+1)\,(d \geq 2)$. *If* $|\det A| < 1$, *and* $\{M_\lambda \eta_0 : \lambda \in A\mathbb{Z}^{2d}\}$ *is a Riesz sequence in* $L^2$, *then the associated operator class* $\mathcal{H} = \{H : \eta_H \in \overline{\operatorname{span}}\, M_\lambda \eta_0\}$ *is not identifiable.*

*Proof.* We set $\lambda = (x_\lambda, \omega_\lambda)$ for $\lambda \in A\mathbb{Z}^{2d}$. The condition $\{M_\lambda \eta_0 : \lambda \in \Lambda\}$ is a Riesz sequence in $L^2$ ensures that the operator

$$D : \ell_0(\mathbb{Z}^{2d}) \to \mathcal{H}, \tag{4.37}$$

$$D : \{c_\lambda\} \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) H_0 \pi(\lambda)^{-1} = \sum_{\lambda\in\Lambda} c_\lambda T_{x_\lambda} M_{\omega_\lambda} H_0 M_{-\omega_\lambda} T_{-x_\lambda} \tag{4.38}$$

is bounded and invertible. We use the argument as already outlined in Proposition 4.10 but must make several changes.

To define a bounded and invertible analysis operator $C : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^{2d})$, we use the tensor product of normalized one-dimensional Gaussians $\gamma_1$ and the fact that we can choose some $u$ such that $1 < u^{-2} < |\det A|^{-1}$ so that $u^2 < 1$. Then we know that the Gabor system $(\gamma_1, u\mathbb{Z}^d \times u\mathbb{Z}^d)$ is a frame for $L^2(\mathbb{R}^d)$ [Lyu92]. Hence the analysis map

$$C : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^{2d}) : \quad C : f \mapsto \{\langle f, M_{ul'} T_{uk'} \gamma_1 \rangle\}_{k',l' \in \mathbb{Z}^d}$$

is well-defined, bounded and invertible. We let $\mathbf{z}' = (k', l')$.

Then we shall examine the composition map

$$C \circ \Phi_f \circ D : \{c_\lambda\} \mapsto \{\langle D(c_\lambda)f, M_{ul'}T_{uk'}\gamma_1\rangle\}.$$

It can be represented as a matrix acting on $\mathbf{c} = \{c_\lambda\}$ in the following way,

$$M\mathbf{c} := (C \circ \Phi_f \circ D(c_\lambda))_{\mathbf{z}'} = \sum_\lambda c_\lambda m_{\mathbf{z}';\lambda}$$

where $M$ has entry coefficients

$$m_{\mathbf{z}';\lambda} = \langle M_{\omega_\lambda}T_{x_\lambda}H_0T_{-x_\lambda}M_{-\omega_\lambda}f, M_{ul'}T_{uk'}\gamma_1\rangle \tag{4.39}$$

To make estimates in the matrix coefficients defined analogously to (4.34), we relabel the lattice points $\lambda = (x, \omega) \in A\mathbb{Z}^{2d}$, by a vector $\mathbf{z} \in \mathbb{Z}^{2d}$, where $\mathbf{z} := A^{-1}\lambda$. Hence, after substituting in (4.35) we obtain

$$\begin{aligned} |m_{\mathbf{z}',\mathbf{z}}| &= |\langle M_{\omega_\lambda}T_{x_\lambda}H_0T_{-x_\lambda}M_{-\omega_\lambda}f, M_{ul'}T_{uk'}\gamma_1\rangle| \\ &\le \langle T_{x_\lambda}|H_0T_{-x_\lambda}M_{-\omega_\lambda}f|, T_{uk'}|\gamma_1|\rangle \\ &= |H_0T_{-x_\lambda}M_{-\omega_\lambda}f| * \gamma_1(x_\lambda - uk') \\ &\le \varphi_1 * \gamma_1(x_\lambda - uk') \cdot \|T_{-x_\lambda}M_{-\omega_\lambda}f\|_{S_0'} \end{aligned} \tag{4.40}$$

and after substituting in (4.36)

$$\begin{aligned} |m_{\mathbf{z}',\mathbf{z}}| &= |\langle T_{\omega_\lambda}M_{-x_\lambda}(\mathcal{F}H_0T_{-x_\lambda}M_{-\omega_\lambda}f), T_{ul'}M_{-uk'}\gamma_1\rangle| \\ &\le \langle T_{\omega_\lambda}|\mathcal{F}H_0T_{-x_\lambda}M_{-\omega_\lambda}f|, T_{ul'}|\gamma_1|\rangle \\ &= |\mathcal{F}H_0T_{-x_\lambda}M_{-\omega_\lambda}f| * \gamma_1(\omega_\lambda - ul') \\ &\le \varphi_2 * \gamma_1(\omega_\lambda - ul') \cdot \|T_{-x_\lambda}M_{-\omega_\lambda}f\|_{S_0'} \end{aligned} \tag{4.41}$$

We can simplify the expressions for the $S_0$-norms in (4.40) and (4.41) because

$$\|T_{-x_\lambda}M_{-\omega_\lambda}f\|_{S_0'} = \|f\|_{S_0'}$$

Then we define

$$h(y) = \max\{\varphi_1 * \gamma_1(y), \varphi_1 * \gamma_1(-y), \varphi_2 * \gamma_1(y), \varphi_2 * \gamma_1(-y)\} \cdot \|f\|_{S_0'}.$$

After combining the two bounds for $|m_{\mathbf{z}',\mathbf{z}}|$, we have

$$|m_{\mathbf{z}',\mathbf{z}}| \le h(\max\{|x_\lambda - uk'|, |\omega_\lambda - ul'|\}) = h(\|\mathbf{z} - u\mathbf{z}'\|_\infty).$$

Now we are again able to apply Lemma 4.7 to obtain a contradiction that $M$ does not have a (bounded) inverse. Hence, we conclude that the operator family is not identifiable. □

| Case | $\Lambda$ | $\Lambda' = \tilde{B}\tilde{C} \cdot \Lambda$ | $D_2(\Lambda)$ |
|---|---|---|---|
| B2 | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha & 0 \\ 0 & \beta \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha & 0 \\ 0 & \beta\sqrt{2} \end{pmatrix}$ | $\frac{1}{\lvert\alpha\beta\rvert}$ |
| B6 | $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & \beta \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ 0 & 0 \\ 0 & \beta\sqrt{2} \\ \alpha & 0 \end{pmatrix}$ | $\frac{1}{\lvert\alpha\beta\rvert}$ |
| B10 | $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \beta \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \\ \alpha & \beta\sqrt{2} \end{pmatrix}$ | $\frac{1}{\lvert\alpha\beta\rvert}$ |
| E1 | $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \\ 0 & \beta \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ 0 & \beta/\sqrt{2} \\ 0 & \beta(1+\sqrt{2}) \\ \alpha & 0 \end{pmatrix}$ | $\frac{1}{\lvert\alpha\beta\rvert\sqrt{2}}$ |
| E2 | $\begin{pmatrix} \alpha & 0 \\ \beta & 0 \\ 0 & \beta \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ \beta/\sqrt{2} & 0 \\ \beta & \beta\sqrt{2} \\ \alpha & 0 \end{pmatrix}$ | $\frac{1}{\lvert\beta\rvert\sqrt{\alpha^2+\beta^2}}$ |
| E4 | $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \\ 0 & 0 \\ 0 & \beta \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ 0 & \beta/\sqrt{2} \\ 0 & \beta \\ \alpha & \beta\sqrt{2} \end{pmatrix}$ | $\frac{1}{\lvert\alpha\beta\rvert\sqrt{2}}$ |
| E5 | $\begin{pmatrix} \alpha & 0 \\ \beta & 0 \\ 0 & 0 \\ 0 & \beta \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ \beta/\sqrt{2} & 0 \\ \beta & 0 \\ \alpha & \beta\sqrt{2} \end{pmatrix}$ | $\frac{1}{\lvert\beta\rvert\sqrt{\alpha^2+\beta^2}}$ |
| F2 | $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \\ \beta & 0 \\ 0 & \alpha \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ 0 & \beta/\sqrt{2} \\ \beta\sqrt{2} & \beta \\ \alpha & \alpha\sqrt{2} \end{pmatrix}$ | $\frac{1}{\sqrt{\alpha^2+\beta^2}}$ |
| F3 | $\begin{pmatrix} \alpha & 0 \\ \beta & 0 \\ 0 & \beta \\ 0 & \alpha \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ \beta/\sqrt{2} & 0 \\ \beta & \beta\sqrt{2} \\ \alpha & \alpha\sqrt{2} \end{pmatrix}$ | $\frac{1}{\sqrt{\alpha^2+\beta^2}}$ |
| F5 | $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \\ 0 & \alpha \\ \beta & 0 \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ 0 & \beta/\sqrt{2} \\ 0 & \beta+\alpha\sqrt{2} \\ \alpha+\beta\sqrt{2} & 0 \end{pmatrix}$ | $\frac{1}{\sqrt{\alpha^2+\beta^2}}$ |
| F6 | $\begin{pmatrix} \alpha & 0 \\ \beta & 0 \\ 0 & \alpha \\ 0 & \beta \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ \beta/\sqrt{2} & 0 \\ \beta & \alpha\sqrt{2} \\ \alpha & \beta\sqrt{2} \end{pmatrix}$ | $\frac{1}{\sqrt{\alpha^2+\beta^2}}$ |
| G1 | $\begin{pmatrix} 0 & 0 \\ \beta & 0 \\ \beta & 0 \\ 0 & \alpha \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ \beta/\sqrt{2} & 0 \\ \beta(1+\sqrt{2}) & 0 \\ 0 & \alpha\sqrt{2} \end{pmatrix}$ | $\frac{1}{\lvert\alpha\beta\rvert\sqrt{2}}$ |
| G3 | $\begin{pmatrix} 0 & 0 \\ \beta & 0 \\ 0 & \beta \\ \alpha & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ \beta/\sqrt{2} & 0 \\ \beta & \beta\sqrt{2} \\ \alpha\sqrt{2} & 0 \end{pmatrix}$ | $\frac{1}{\lvert\beta\rvert\sqrt{\alpha^2+\beta^2}}$ |
| G4 | $\begin{pmatrix} 0 & 0 \\ \beta & 0 \\ 0 & \alpha \\ \beta & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ \beta/\sqrt{2} & 0 \\ \beta & \alpha\sqrt{2} \\ \beta\sqrt{2} & 0 \end{pmatrix}$ | $\frac{1}{\lvert\alpha\beta\rvert\sqrt{2}}$ |
| G6 | $\begin{pmatrix} 0 & 0 \\ \beta & 0 \\ \alpha & 0 \\ 0 & \beta \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ \beta/\sqrt{2} & 0 \\ \alpha\sqrt{2}+\beta & 0 \\ 0 & \beta\sqrt{2} \end{pmatrix}$ | $\frac{1}{\lvert\beta\rvert\sqrt{\alpha^2+\beta^2}}$ |
| H1 | $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & \beta \\ 0 & \alpha \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ 0 & 0 \\ 0 & \beta\sqrt{2} \\ \alpha & \alpha\sqrt{2} \end{pmatrix}$ | $\frac{1}{\lvert\alpha\rvert\sqrt{\alpha^2+\beta^2}}$ |
| H2 | $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \\ \beta & 0 \\ 0 & \alpha \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ 0 & 0 \\ \beta\sqrt{2} & 0 \\ \alpha & \alpha\sqrt{2} \end{pmatrix}$ | $\frac{1}{\lvert\alpha\rvert\sqrt{\alpha^2+\beta^2}}$ |
| H5 | $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & \alpha \\ \beta & 0 \end{pmatrix}$ | $\begin{pmatrix} \alpha/\sqrt{2} & 0 \\ 0 & 0 \\ 0 & \alpha\sqrt{2} \\ \alpha+\beta\sqrt{2} & 0 \end{pmatrix}$ | $\frac{1}{\lvert\alpha\rvert\sqrt{\alpha^2+\beta^2}}$ |

## 4.6 Case study $\kappa_0(x,\omega) = e^{-\pi(x^2+\omega^2)}$

In this section we consider some of the operator families from Tables 2 and 3 when the kernel of the prototype operator $H_0$ is $\kappa_0(x,\omega) = e^{-\pi(x^2+\omega^2)}$, $(x,\omega) \in \mathbb{Z}^2$. We will vary the lattice $\Lambda$ used in the construction of the operator family $\mathcal{H}_\Lambda$. For every index set $\Lambda$ we shall search for criteria such that $\mathcal{H}_\Lambda$ becomes identifiable. That will naturally include conditions which make the identification problem well-posed.

In this case associated spreading function $\eta_0(t,\nu)$ is given by

$$\begin{aligned} \eta_0(t,\nu) &= \int \kappa_0(x, x-t) e^{-2\pi i \nu x} dx \\ &= \int e^{-\pi(2x^2-2xt+t^2)} e^{-2\pi i \nu x} dx \\ &= e^{-\frac{\pi}{2}t^2} \int e^{-\pi(x\sqrt{2}-\frac{t}{\sqrt{2}})^2} e^{-2\pi i \nu x} dx \\ &= \frac{1}{\sqrt{2}} e^{-\frac{\pi}{2}t^2} M_{-\frac{t}{\sqrt{2}}}(e^{-\frac{\pi}{2}\nu^2}) \\ &= \frac{1}{\sqrt{2}} e^{-\pi i \sqrt{2} t\nu} e^{-\frac{\pi}{2}(t^2+\nu^2)}. \end{aligned} \tag{4.42}$$

In other words, $\eta_0$ is the image of the standard 2-dimensional Gaussian $\gamma_2(t,\nu) = e^{-\pi(t^2+\nu^2)}$ under a symplectic transformation, which we write as

$$\eta_0 = \mu(\tilde{C})^{-1} \mu(\tilde{B})^{-1} \gamma_2.$$

Following [Fol89], the symplectic operators $\mu(\tilde{B})$ (a dilation), $\mu(\tilde{C})$ (a chirp) are associated respectively to the lattices

$$\tilde{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 1 \end{pmatrix}.$$

The spreading function $\eta_0$ therefore belongs to $\mathcal{S}(\mathbb{R}^2)$ as well.

We shall consider different choices for the lattice $\Lambda$ for the Gabor Riesz basis sequence $(\eta_0, \Lambda)$. The examples in this section follow the following pattern. Since $\eta_0$ is fixed and we vary the lattice $\Lambda$, we must observe that condition (III) holds (namely $(\eta_0, \Lambda)$ must be a Riesz basis sequence in $L^2(\mathbb{R}^2)$ in order for our approach make sense). To verify this, we apply Theorem 2.2 to transform this question to known results for Gabor Riesz basis sequences based on the standard Gaussian. Thus we have to show in each case that $(\gamma_2, \tilde{B}\tilde{C} \cdot \Lambda)$ is a Riesz basis sequence for the lattices $\Lambda' = \tilde{B}\tilde{C} \cdot \Lambda$. The matrix $\tilde{B}\tilde{C}$ is

$$\tilde{B}\tilde{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 1 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

Table 4 lists some choices of $\Lambda$ together with the lattice $\Lambda' = \tilde{B}\tilde{C} \cdot \Lambda$ and the 2-density of $\Lambda, D_2(\Lambda)$. We examine each case separately. In the following propositions we shall drop the subscript whenever it is clear which lattice we refer to. To prove identifiability, all we have to show is that under the assumptions in each case both sequences: $(\gamma_2, \Lambda')$ and $\{H_\lambda f : \lambda \in \Lambda\}$ are Riesz basis sequences.

**Proposition 4.14 (Case B2)** *Let* $\eta_0$ *be given by* (4.42)*, and* $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha & 0 \\ 0 & \beta \end{pmatrix} \mathbb{Z}^2$ *(see Table 4). If* $D_2(\Lambda) < 1$*, the operator family* $\mathcal{H}_{B2} = \{H : \eta_H \in \overline{\operatorname{span}}(\eta_0, \Lambda)\}$ *is identifiable.*

*Proof.* The condition $D_2(\Lambda) < 1$ is equivalent to $|\alpha\beta| > 1$, see Table 4. In case B2 we verify the Riesz basis sequence condition for $(\eta_0, \Lambda)$ on the set of spreading functions by checking whether the Gabor system $(\gamma_2, \Lambda')$ is a Riesz basis sequence. In this case

$$\Lambda' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha & 0 \\ 0 & \beta\sqrt{2} \end{pmatrix} \begin{pmatrix} n \\ l \end{pmatrix}$$

which is a tensor system with respect to $\gamma_2$. Furthermore $(\gamma_2, \Lambda')$ is a Riesz basis sequence, which can be deduced from Theorem 7.2.3, [Chr03]. We choose as identifier $f = \delta_{\frac{\mathbb{Z}}{\beta}}$ and see that

$$M_{\beta l} T_{-\alpha n} H_0 T_{\alpha n} M_{-\beta l} f = M_{\beta l} T_{-\alpha n} H_0 T_{\alpha n} \delta_{\frac{\mathbb{Z}}{\beta}}$$

Furthermore,

$$H_0 T_{\alpha n} \delta_{\frac{\mathbb{Z}}{\beta}} = e^{-\pi x^2} \sum_j e^{-\pi(\frac{j}{\beta} - \alpha n)^2}.$$

We denote the quantity

$$C(n) = \sum_j e^{-\pi(\frac{j}{\beta} - \alpha n)^2}, \quad C(n) > 0, \forall n.$$

and check that the sequence $\{h_{l,n} : l, n \in \mathbb{Z}\}$, where

$$h_{l,n} = C(n) M_{\beta l} T_{-\alpha n} \gamma_1 : l, n \in \mathbb{Z}\},$$

is a Riesz basis sequence if $|\beta\alpha| > 1$.

We use the criterion for Riesz basis sequences given in Lemma 3.6.2, [Chr03]. The sequence $(\gamma_1, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a Riesz basis whenever $|\beta\alpha| > 1$ [Lyu92], [SW92], hence it is an unconditional basis[1]. Since each element of $\{h_{k,l} : k, l \in \mathbb{Z}\}$ is a scalar multiple of an element from an unconditional basis, not only does $\{h_{k,l} : k, l \in \mathbb{Z}\}$ span the same subspace of $L^2(\mathbb{R})$, but it is also an unconditional basis for this subspace.

[1] An unconditional basis is a basis with the additional property that the convergence of the basis expansion is unconditional (i.e. convergence does not depend on the order of summation).

To complete the proof we must finds bounds on $\|h_{l,n}\|_2$, in other words, show that $0 < \inf_{n\in\mathbb{Z}} C(n) \leq \sup_{n\in\mathbb{Z}} C(n) < \infty$. We observe that namely, for a fixed $n$, there exists an index $j'$ such that $\frac{j'}{\beta} \leq \alpha n \leq \frac{j'+1}{\beta}$, hence $|\frac{j}{\beta} - \alpha n| \geq \frac{1}{\beta}$ for all $j$. Thus $C(n) \geq e^{-\frac{\pi}{\beta^2}}$. To make the upper estimate we use the property that $e^{-x^2}$ is monotonic in $(-\infty, 0)$ and $(0, \infty)$. Without loss of generality, when $\beta > 0$, for all $j \leq j'$,

$$e^{-\pi\left(\frac{j}{\beta}-\alpha n\right)^2} \leq e^{-\pi\left(\frac{j'-j}{\beta}\right)^2},$$

and for all $j \geq j'+1$,

$$e^{-\pi\left(\frac{j}{\beta}-\alpha n\right)^2} \leq e^{-\pi\left(\frac{j'-j+1}{\beta}\right)^2}.$$

Therefore,

$$C(n) \leq \sum_{j\leq j'} e^{-\pi\left(\frac{j'-j}{\beta}\right)^2} + \sum_{j\geq j'+1} e^{-\pi\left(\frac{j'-j+1}{\beta}\right)^2} = C(0) < \infty.$$

Thus the system $\{h_{l,n} : l, n \in \mathbb{Z}\}$ is an unconditional basis and

$$0 < \inf_{l,n\in\mathbb{Z}} \|h_{l,n}\|_2 \leq \sup_{l,n\in\mathbb{Z}} \|h_{l,n}\|_2 < \infty,$$

hence it is a Riesz basis sequence.

Thus, by proving that $(\gamma_2, \Lambda')$ and $\{C(n)M_{\beta l}T_{-\alpha n}\gamma_1 : n, l \in \mathbb{Z}\}$ are Riesz basis sequences, we have in fact shown that for $f = \delta_{\frac{\mathbb{Z}}{\beta}}$, and $\Phi_f : H \mapsto Hf$, the norm equivalence

$$\|\Phi_f H\|_{L^2} \asymp \|H\|_{HS},$$

holds. Therefore, the class of operators $\mathcal{H}_{B2} = \{H : \eta_H \in \overline{\text{span}}\,(\eta_0, \Lambda)\}$ is identifiable. □

**Proposition 4.15 (Case B6)** *Let $\eta_0$ be given by (4.42), and $\Lambda_{B6} = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & \beta \\ 0 & 0 \end{pmatrix} \mathbb{Z}^2$ (Table 4). If $D_2(\Lambda_{B6}) < 1$, then the operator family $\mathcal{H}_{B6} = \{H : \eta_H \in \overline{\text{span}}\,(\eta_0, \Lambda_{B6})\}$, is identifiable.*

*Proof.* The condition $D_2(\Lambda_{B6}) < 1$ is equivalent to $|\alpha\beta| > 1$. We first verify that $(\eta_0, \Lambda_{B6})$ is a Riesz basis sequence (condition (III)) for $\eta_0$ given in (4.42). By Theorem 2.2, $(\eta_0, \Lambda_{B6})$ being a Riesz basis sequence is equivalent to $(\gamma_2, \Lambda'_{B6})$ being a Riesz basis sequence, where

$$\Lambda' = \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ 0 & 0 \\ 0 & \beta\sqrt{2} \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \subset \frac{\alpha}{\sqrt{2}}\mathbb{Z} \times 0 \times \beta\sqrt{2}\mathbb{Z} \times \alpha\mathbb{Z}.$$

If $|\alpha\beta| > 1$, then $(\gamma_1, \frac{\alpha}{\sqrt{2}}\mathbb{Z} \times \beta\sqrt{2}\mathbb{Z})$ is a Riesz sequence. Hence, $(\gamma_2, \Lambda'_{B6})$ is a Riesz sequence. The remainder of the proof follows the line of Proposition 4.14. □

We give another example about the operator family $\mathcal{H}_{B6}$. For a different choice of $\eta_0$ it is identifiable as well.

**Corollary 4.16 (Case B6)** *Let* $\eta_0 = \chi_{[0,\frac{1}{\alpha})} \otimes \chi_{[0,\frac{1}{\beta})}$. *If* $|\alpha\beta| \geq 1$, *then the operator family*

$$\mathcal{H}_{B6} = \{H : \eta_H \in \overline{\text{span}}\,(\eta_0, \Lambda_{B6}) \cap M^1\}$$

*is identifiable with* $f = \delta_{\alpha\mathbb{Z}}$.

*Proof.* Observe that this identifier $f = \delta_{\alpha\mathbb{Z}} \in M^\infty$. Due to $\alpha$-periodicity in fact $T_{\alpha k} f = f$ for all $k \in \mathbb{Z}$. The action of the prototype operator $H_{k,l} = e^{-2\pi i\beta l\cdot\alpha k} M_{\beta l} T_{\alpha k} H_0 M_{-\beta l}$ (see B6 in Table 2) on $f$ can be rewritten as follows

$$\begin{aligned} H_{k,l} f &= e^{-2\pi i\beta l\cdot\alpha k} M_{\beta l} T_{\alpha k} H_0 M_{-\beta l} f \\ &= e^{-2\pi i\beta l\cdot\alpha k} M_{\beta l} T_{\alpha k} H_0 M_{-\beta l} T_{-\alpha k} f \\ &= M_{\beta l} T_{\alpha k} H_0 T_{-\alpha k} M_{-\beta l} f \end{aligned}$$

Thus $\{H_{k,l} f\}$ is the same family of functions as in those in case B2, discussed in detail in [KP06], but we have substituted a specific $f$, which is a periodic delta-train. Furthermore, each spreading function $\eta_H$ is supported on $[0, \frac{1}{\alpha}) \times [0, \frac{1}{\beta})$ and has a canonical ONB series expansion

$$\eta_H(t,\nu) = \sum_{k,l\in\mathbb{Z}} c_{k,l} e^{2\pi i(\alpha k t + \beta l \nu)}$$

because

$$\eta_0 = \chi_{[0,\frac{1}{\alpha})} \otimes \chi_{[0,\frac{1}{\beta})}.$$

This ensures that every $H \in \mathcal{H}$ has an ONB expansion in terms of $H_{k,l}$. For the rest, this case is equivalent to the one described in Theorem 3.1, [KP05]. □

For the examples that follow, we use a lemma about Gaussian Riesz basis sequences.

**Lemma 4.17** *Let* $a, b \neq 0$. *Then* $\{T_{an} M_{bn} \gamma_1, n \in \mathbb{Z}\}$ *is a Riesz basis sequence.*

*Proof.* Note that if $b = 0$, the theory from [Chr03] (Theorem 7.2.3) applies because the function

$$\Phi_{\gamma_1}(\xi) = \sum_{n\in\mathbb{Z}} |\widehat{\gamma}_1(\tfrac{\xi+n}{a})|^2 \quad (\text{compare [Chr03] : (7.2)})$$

is bounded above and away from 0 on $[0,1)$. If $a = 0$, then $\mathcal{F} M_{bn} \gamma_1 = T_{bn} \gamma_1$. To prove our claim we note that $\Lambda = \{(an, bn), n \in \mathbb{Z}\}$ is symplectically equivalent to $\Lambda' = \{(an, 0), n \in \mathbb{Z}\}$ via $M = \begin{pmatrix} 1 & 0 \\ -\frac{b}{a} & 1 \end{pmatrix}$. Theorem 2.2 shows that $(\gamma_1, \Lambda)$ is a Riesz basis sequence if and only if $(\mu(M)\gamma_1, \Lambda')$ is a Riesz basis sequence, where

$$\mu(M)\gamma_1(t) = e^{\pi i \frac{b}{a} t^2} e^{-\pi t^2}.$$

We apply Theorem 7.2.3 from [Chr03] to the system of translates $\{T_{an}\mu(M)\gamma_1 : n \in \mathbb{Z}\}$. The associated function is

$$\Phi_{\mu(M)\gamma_1}(\xi) = \sum_{n\in\mathbb{Z}} |\mathcal{F}\mu(M)\gamma_1(\tfrac{\xi+n}{a})|^2,$$

but

$$\mathcal{F}\mu(M)\gamma_1(\xi) = \int e^{\pi i \frac{b}{a} t^2} e^{-\pi t^2} e^{-2\pi i t\xi} dt = \frac{1}{(1-i)^{\frac{1}{2}}} e^{-\pi\xi^2(\frac{1+i}{2})}.$$

Hence

$$\begin{aligned}
\Phi_{\mu(M)\gamma_1}(\xi) &= \frac{1}{|1-i|} \sum_{n\in\mathbb{Z}} |e^{-\pi\left(\frac{\xi+n}{a}\right)^2(\frac{1+i}{2})}|^2 \\
&= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} |e^{-\pi\left(\frac{\xi+n}{a}\right)^2(\frac{1+i}{2})}|^2 \\
&= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} |e^{-\frac{1}{2}\pi\left(\frac{\xi+n}{a}\right)^2}|^2 \cdot \underbrace{|e^{-\frac{i}{2}\pi\left(\frac{\xi+n}{a}\right)^2}|^2}_{1} \\
&= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} |e^{-\pi\left(\frac{\xi+n}{a}\right)^2}|^2
\end{aligned} \tag{4.43}$$

Since the Gaussian $\gamma_1 \in \mathcal{S}(\mathbb{R}) \cap L^1(\mathbb{R})$, the expression for $\Phi_{\mu(M)\gamma_1}(\xi)$ is bounded above and away from zero on $[0,1)$. Thus Theorem 7.2.3 from [Chr03] shows that $\{T_{an}\mu(M)\gamma_1 : n \in \mathbb{Z}\}$ is a Riesz basis sequence. □

**Proposition 4.18 (Case D5)** *Let* $\eta_0$ *be given by* (4.42), *and* $\Lambda = \begin{pmatrix} \alpha & 0 \\ \beta & 0 \\ 0 & \alpha \\ 0 & 0 \end{pmatrix} \mathbb{Z}^2$ *(Table 4). If* $|\alpha| > 1$, *then the operator family* $\mathcal{H}_{D5} = \{H : \eta_H \in \overline{\operatorname{span}}(\eta_0, \Lambda_{D5})\}$ *is identifiable.*

*Proof.* We follow the strategy of the previous examples. We begin the proof by checking that $(\eta_0, \Lambda)$ is a Riesz basis sequence and use symplectic equivalence (Theorem 2.2). The lattice $\Lambda'$ is given by

$$\Lambda' = \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \frac{\beta}{\sqrt{2}} & 0 \\ \beta & \alpha\sqrt{2} \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix}.$$

Therefore, the Gabor system $(\gamma_2, \Lambda')$ is a tensor system $(\gamma_1, \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \beta & \alpha\sqrt{2} \end{pmatrix} \mathbb{Z}^2) \times (\gamma_1, \begin{pmatrix} \frac{\beta}{\sqrt{2}} \\ \alpha \end{pmatrix} \mathbb{Z})$, which is a Riesz basis sequence if $|\alpha| > 1$ (a criterion coming from the first component); for the second component see Lemma 4.17.

We use as identifier $f = \delta_0$. Thus

$$H_{k,l} f = T_{\alpha k} M_{\alpha l} H_0 M_{\beta k - \alpha l} f = T_{\alpha k} M_{\alpha l} \gamma_1,$$

because

$$H_0 M_{\beta k-\alpha l} f(x) = \gamma_1(x) \int_{\mathbb{R}} \gamma_1(x-t)(M_{\beta k-\alpha l}\delta_0)(x-t)dt = \gamma_1(x).$$

Hence, the sequence $\{H_{k,l}\}_{k,l\in\mathbb{Z}}$ is a Riesz basis sequence if and only if $|\alpha| > 1$ [Lyu92]. □

*Note*: The 2-density of $\Lambda_{D5}$ equals $\frac{1}{|\alpha|\sqrt{\alpha^2+\beta^2}}$, which is less than 1 if $|\alpha| > 1$, *but not vice versa*! For identification of $\mathcal{H}_{D5}$, the condition $|\alpha| > 1$ is stronger than the 2-density condition.

**Proposition 4.19 (Case D6)** *Let* $\eta_0$ *be given by* (4.42)*, and* $\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \\ 0 & \alpha \\ 0 & 0 \end{pmatrix} \mathbb{Z}^2$ *(Table 4). If* $\alpha, \beta$ *are such that* $|\alpha(\beta + \alpha\sqrt{2})| \geq \sqrt{2}, |\alpha\beta| > \sqrt{2}, |\alpha| > 1$*, then the operator family* $\mathcal{H}_{D6} = \{H : \eta_H \in \overline{\text{span}}\,(\eta_0, \Lambda)\}$ *is identifiable.*

*Proof.* This case is again similar to the previous ones. We have to check that $(\eta_0, \Lambda)$ is a Riesz basis sequence. We use symplectic equivalence (Theorem 2.2) and consider instead the Gabor system $(\gamma_2, \Lambda')$, where

$$\Lambda' = \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ 0 & \frac{\beta}{\sqrt{2}} \\ 0 & \beta+\alpha\sqrt{2} \\ \alpha & 0 \end{pmatrix} \mathbb{Z}^2.$$

The associated Gabor system $(\gamma_2, \Lambda')$ is a tensor Riesz sequence if $|\alpha\beta| > \sqrt{2}$ and $|\alpha(\beta + \alpha\sqrt{2})| \geq \sqrt{2}$.

We choose as identifier $f = \delta_0$. As in Proposition 4.18 we have that

$$H_{k,l} f = T_{\alpha k} M_{\alpha l} H_0 M_{(\beta-\alpha)l} f = T_{\alpha k} M_{\alpha l} \gamma_1.$$

Hence the sequence $\{H_{k,l}\}_{k,l\in\mathbb{Z}}$ is a Riesz basis sequence if and only if $|\alpha| > 1$ [Lyu92]. □

*Note*: In case D6, the 2-density of $\Lambda$ is $D_2(\Lambda) = \frac{1}{|\alpha|\sqrt{\alpha^2+\beta^2}}$. We have three conditions on the parameters $\alpha, \beta$: $|\alpha(\beta + \alpha\sqrt{2})| \geq \sqrt{2}, |\alpha\beta| > \sqrt{2}, |\alpha| > 1$, which are illustrated in Figure 1.

**Proposition 4.20 (Case E1)** *Let* $\eta_0$ *be given by* (4.42)*, and* $\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \\ 0 & \beta \\ 0 & 0 \end{pmatrix} \mathbb{Z}^2$ *(Table 4). If* $D_2(\Lambda) < \frac{1}{2}$ *(in other words* $|\alpha\beta| > \sqrt{2}$*), then the operator family* $\mathcal{H}_{E1} = \{H : \eta_H \in \overline{\text{span}}\,(\eta_0, \Lambda_{E1})\}$ *is identifiable.*

*Proof.* Essentially similar to the previous cases. We begin the proof by checking that $(\eta_0, \Lambda)$ is a Riesz basis sequence and use symplectic equivalence (Theorem 2.2). The Gabor system $(\gamma_2, \Lambda')$ is a Riesz basis sequence if $|\alpha\beta| > \sqrt{2}$, because

$$\Lambda' = \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ 0 & \frac{\beta}{\sqrt{2}} \\ 0 & \beta(1+\sqrt{2}) \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \subset \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ 0 & \beta(1+\sqrt{2}) \end{pmatrix} \mathbb{Z}^2 \times \begin{pmatrix} 0 & \frac{\beta}{\sqrt{2}} \\ \alpha & 0 \end{pmatrix} \mathbb{Z}^2$$

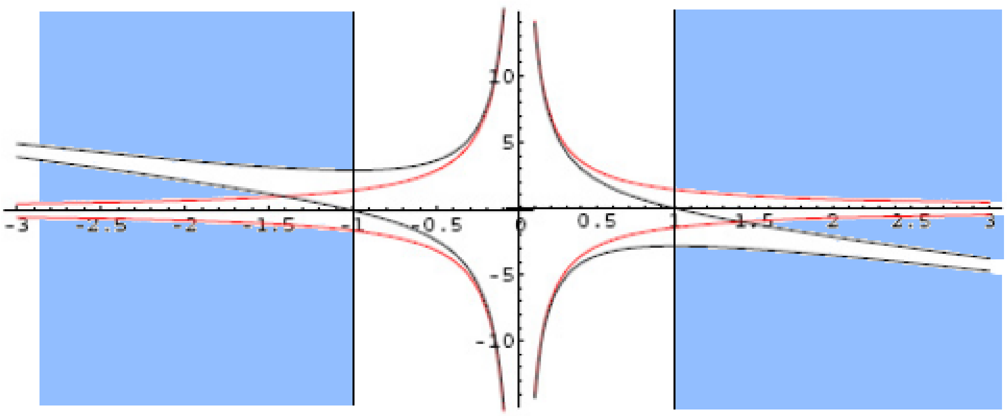


Figure 1: The set $(\alpha, \beta)$ fulfilling the conditions in case D6 lies in the blue region.

and the system $(\gamma_2, \left(\begin{smallmatrix}\frac{\alpha}{\sqrt{2}} & 0\\ 0 & \beta(1+\sqrt{2})\end{smallmatrix}\right)\mathbb{Z}^2 \times \left(\begin{smallmatrix}0 & \frac{\beta}{\sqrt{2}}\\ \alpha & 0\end{smallmatrix}\right)\mathbb{Z}^2)$ is a tensor Riesz basis sequence under the given assumptions (since $|\alpha\beta|\frac{1+\sqrt{2}}{\sqrt{2}}, \frac{|\alpha\beta|}{\sqrt{2}} > 1$). Hence, $(\gamma_2, \Lambda')$ is a Riesz basis sequence, and furthermore, according to Theorem 2.2 $(\eta_0, \Lambda)$ is also a Riesz basis sequence. We use as identifier $f = \delta_{\mathbb{Z}}$ and check that $\{H_{k,l}f\}$ is a Riesz basis sequence.

$$H_{k,l}f = M_{\beta l}T_{\alpha k}H_0 f = M_{\beta l}T_{\alpha k}e^{-\pi x^2}\sum_{j\in\mathbb{Z}} e^{-\pi j^2}, \tag{4.44}$$

$\{H_{k,l}f\}$ is also Riesz basis sequence if $|\alpha\beta| > 1$ because the expression on the right-hand side of (4.44) is a constant multiple of the Riesz basis sequence $(\gamma_1, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ [Lyu92], [SW92]. This is a consequence of the fact that $\sum_{j\in\mathbb{Z}^d} e^{-\pi j^2}$ is a nonzero constant. Hence, the family $\{M_{\beta l}T_{\alpha k}H_0 f : k, l \in \mathbb{Z}\}$ is a Riesz basis sequence if $|\alpha\beta| > \sqrt{2} > 1$. □

**Proposition 4.21 (Case E2)** *Let $\eta_0$ be given by (4.42), and $\Lambda = \left(\begin{smallmatrix}\alpha & 0\\ \beta & 0\\ 0 & \beta\\ 0 & 0\end{smallmatrix}\right)\mathbb{Z}^2$ (Table 4). If $|\alpha\beta| > 1$, then the operator family $\mathcal{H}_{E2} = \{H : \eta_H \in \overline{\operatorname{span}}(\eta_0, \Lambda_{E2})\}$ is identifiable.*

*Proof.* The line follows the line of proof of Proposition 4.20. We first demonstrate that $(\eta_0, \Lambda)$ is a Riesz basis sequence with the help of Theorem 2.2. We show equivalently that the Gabor system $(\gamma_2, \Lambda')$ is a Riesz basis sequence. But

$$\Lambda' = \begin{pmatrix}\frac{\alpha}{\sqrt{2}} & 0\\ \frac{\beta}{\sqrt{2}} & 0\\ \beta & \beta\sqrt{2}\\ \alpha & 0\end{pmatrix}\begin{pmatrix}k\\ l\end{pmatrix} \subset \begin{pmatrix}\frac{\alpha}{\sqrt{2}} & 0\\ \beta & \beta\sqrt{2}\end{pmatrix}\mathbb{Z}^2 \times \begin{pmatrix}\frac{\beta}{\sqrt{2}} & 0\\ \alpha & 0\end{pmatrix}\mathbb{Z}^2.$$

The lattice

$$\begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \beta & \beta\sqrt{2} \end{pmatrix} \mathbb{Z}^2 \times \begin{pmatrix} \frac{\beta}{\sqrt{2}} & 0 \\ \alpha & 0 \end{pmatrix} \mathbb{Z}^2,$$

is tensor so its associated Gabor system is a tensor system

$$(\gamma_1, \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \beta & \beta\sqrt{2} \end{pmatrix} \mathbb{Z}^2) \times (\gamma_1, \begin{pmatrix} \frac{\beta}{\sqrt{2}} & 0 \\ \alpha & 0 \end{pmatrix} \mathbb{Z}^2),$$

The last Gabor system is a Riesz basis sequence if and only if $|\alpha\beta| > 1$ (a condition on the first term); for the second term, compare Lemma 4.17. Hence for those values of $(\alpha, \beta)$ its subsequence $(\gamma_2, \Lambda')$ is also a Riesz basis sequence. Thus, $(\eta_0, \Lambda_{E2})$ is a Riesz basis sequence.

As identifier we choose $f = \delta_{\frac{\mathbb{Z}}{\beta}}$[2]. Then

$$H_{k,l} f = T_{\alpha k} M_{\beta l} H_0 M_{\beta(k-l)} f = T_{\alpha k} M_{\beta l} H_0 \delta_{\frac{\mathbb{Z}}{\beta}}$$

We rewrite the expression $H_0 \delta_{\frac{\mathbb{Z}}{\beta}}$ as

$$H_0 \delta_{\frac{\mathbb{Z}}{\beta}} = \langle \delta_{\frac{\mathbb{Z}}{\beta}}, \kappa_0(x, \cdot) \rangle = e^{-\pi x^2} \sum_{j \in \mathbb{Z}^d} e^{-\pi \frac{j^2}{\beta}} \tag{4.45}$$

The sum of the series

$$\sum_{j \in \mathbb{Z}^d} e^{-\pi \frac{j^2}{\beta}}$$

on the right-hand side of (4.45) is some nonzero scalar depending on $\beta$, so $H_0 \delta_{\frac{\mathbb{Z}}{\beta}}$ is a scalar multiple of the Gaussian $\gamma_1 = e^{-\pi x^2}$. If $|\alpha\beta| > 1$, then the Gabor system $(\gamma_1, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a Riesz basis sequence (as seen from previous examples). Therefore, $\{H_{k,l} f : k, l \in \mathbb{Z}\}$ is also a Riesz basis sequence. Hence the operator family $\mathcal{H}_{E2}$ with $\kappa_0(x, \omega) = e^{-\pi(x^2+\omega^2)}$ is identifiable. □

*Note*: The 2-density of the time-frequency lattice in the case E2 (Table 4) is $\frac{1}{|\beta|\sqrt{\alpha^2+\beta^2}}$. The condition $|\alpha\beta| > 1$ is independent of the value of $D_2(\Lambda)$. For instance, $|\alpha\beta| > 1$ implies that $D_2(\Lambda) < 1$, *but not vice versa*! We saw that both $D_2(\Lambda)$ and $|\alpha\beta|$ can be smaller than 1 (choose $\alpha = (10^{10} + 1)^{-1}, \beta = 10^{10}$), and Proposition 4.12 implies that the operator family is not identifiable.

**Proposition 4.22 (Case F2)** *Let* $\eta_0$ *be given by* (4.42)*, and* $\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \\ \beta & 0 \\ 0 & \alpha \end{pmatrix} \mathbb{Z}^2$ *(Table 4). If* $|\alpha\beta| > 1$*, then the operator family* $\mathcal{H}_{F2} = \{H : \eta_H \in \overline{\text{span}}\,(\eta_0, \Lambda_{F2})\}$ *is identifiable.*

[2] Another possibility would be $f = \delta_0$.

*Proof.* The procedure for solving this example is similar to those illustrated in Propositions 4.14-4.21. We verify that the family of spreading functions $(\eta_0, \Lambda_{F2})$ is a Riesz basis sequence with the help of Theorem 2.2. This is equivalent to checking whether the Gabor system $(\gamma_2, \Lambda')$ is a Riesz basis sequence. But we see that the time-frequency index set factorizes,

$$\Lambda' = \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ 0 & \frac{\beta}{\sqrt{2}} \\ \beta\sqrt{2} & \beta \\ \alpha & \alpha\sqrt{2} \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \subset \left(\begin{smallmatrix} \alpha\sqrt{2} & 0 \\ \beta\sqrt{2} & \beta \end{smallmatrix}\right) \mathbb{Z}^2 \times \left(\begin{smallmatrix} 0 & \beta\sqrt{2} \\ \alpha & \alpha\sqrt{2} \end{smallmatrix}\right) \mathbb{Z}^2.$$

Therefore, $(\gamma_2, \Lambda')$ is a Riesz basis sequence if $|\alpha\beta| > 1 > \frac{1}{\sqrt{2}}$.

We choose as identifier $f = \delta_{\frac{\mathbb{Z}}{\beta}}$. Then

$$\begin{aligned} H_{k,m} f &= e^{-2\pi i \beta k \cdot \alpha k} M_{\beta k} T_{\alpha(k-m)} H_0 T_{\alpha m} M_{\beta(m-k)} f \\ &= e^{-2\pi i \beta k \cdot \alpha k} M_{\beta k} T_{\alpha(k-m)} H_0 T_{\alpha m} \delta_{\frac{\mathbb{Z}}{\beta}} \end{aligned} \tag{4.46}$$

The quantity $H_0 T_{\alpha m} \delta_{\frac{\mathbb{Z}}{\beta}}$ in (4.46) can be easily estimated and approximated - we refer to Proposition 4.23. The proof concludes as in Proposition 4.23. □

*Note*: We observe again that in case F2 $|\alpha\beta| > 1$ implies that $D_2(\Lambda) < \frac{1}{\sqrt{2}}$, *but not vice versa*!

**Proposition 4.23 (Case F3)** *Let* $\eta_0$ *given by* (4.42), *and* $\Lambda = \left(\begin{smallmatrix} \alpha & 0 \\ \beta & 0 \\ 0 & \beta \\ 0 & \alpha \end{smallmatrix}\right) \mathbb{Z}^2$ *(see Table 4). If* $|\alpha\beta| > 1$, *then the operator family* $\mathcal{H}_{F3} = \{H : \eta_H \in \overline{\text{span}}(\eta_0, \Lambda)\}$ *is identifiable.*

*Proof.* In case F3 we verify the Riesz sequence condition by using Theorem 2.2. We have to check that the Gabor system $(\gamma_2, \Lambda')$ is a Riesz basis sequence. But in this case

$$\Lambda' = \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \frac{\beta}{\sqrt{2}} & 0 \\ \beta & \beta\sqrt{2} \\ \alpha & \alpha\sqrt{2} \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \beta & \beta\sqrt{2} \end{pmatrix} \mathbb{Z}^2 \times \begin{pmatrix} \frac{\beta}{\sqrt{2}} & 0 \\ \alpha & \alpha\sqrt{2} \end{pmatrix} \mathbb{Z}^2.$$

Hence $(\gamma_2, \Lambda')$ is a tensor system with respect to $\gamma_2$. The criterion in [Lyu92] shows that the above tensor system is a Riesz basis sequence if and only if $|\alpha\beta| > 1$. Hence by symplectic equivalence, $(\eta_0, \Lambda)$ is also a Riesz basis sequence for such $|\alpha\beta| > 1$.

If we choose as identifier $f = \delta_{\frac{\mathbb{Z}}{\beta}}$, then

$$\begin{aligned} H_{k,l} f &= e^{-2\pi i \beta l \cdot \alpha k} M_{\beta l} T_{\alpha(k-l)} H_0 T_{\alpha l} M_{\beta(k-l)} f \\ &= e^{-2\pi i \beta l \cdot \alpha k} M_{\beta l} T_{\alpha(k-l)} H_0 T_{\alpha l} \delta_{\frac{\mathbb{Z}}{\beta}} \end{aligned} \tag{4.47}$$

The expression in (4.47) can be further simplified because

$$H_0 T_{\alpha l} \delta_{\frac{\mathbb{Z}}{\beta}} = e^{-\pi x^2} \sum_{j \in \mathbb{Z}} e^{-\pi(\frac{j}{\beta} - \alpha l)^2}.$$

The quantity given by $\sum_j e^{-\pi(\frac{j}{\beta}-\alpha l)^2}$ will be denoted by $C(l)$. We shall verify that if $\alpha\beta > 1$, then the sequence $\{h_{k,l} : k, l \in \mathbb{Z}\}$, where

$$h_{k,l} = C(l) e^{-2\pi i \beta l \cdot \alpha k} M_{\beta l} T_{\alpha(k-l)} \gamma_1,$$

is a Riesz basis sequence. We follow the steps outlined in Proposition 4.14 and Lemma 3.6.2, [Chr03]. The sequence $\{e^{-2\pi i\beta l\cdot\alpha k} M_{\beta l} T_{\alpha(k-l)}\gamma_1 : k, l \in \mathbb{Z}\}$ is a Riesz basis for $|\alpha\beta| > 1$ [Lyu92], [SW92], hence it is an unconditional basis for its closed linear span. Since each element of $\{h_{k,l} : k, l \in \mathbb{Z}\}$ is a scalar multiple of an element from an unconditional basis, $\{h_{k,l} : k, l \in \mathbb{Z}\}$ spans the same subspace of $L^2(\mathbb{R})$. So $\{h_{k,l} : k, l \in \mathbb{Z}\}$ is also an unconditional basis for its closed linear span.

To prove the Riesz basis property, we must demonstrate that $0 < \inf_l C(l) \leq \sup_l C(l) < \infty$. Observe that for a fixed $l$, there exists an index $j'$ such that $\frac{j'}{\beta} \leq \alpha l \leq \frac{j'+1}{\beta}$, hence $|\frac{j}{\beta} - \alpha l| \geq \frac{1}{\beta}$. Thus $C(l) \geq e^{-\frac{\pi}{\beta^2}}$ for all $l$. For the upper estimate we use the property that $e^{-x^2}$ is monotone in each of the intervals $(-\infty, 0)$ and $(0,\infty)$. Without loss of generality, when $\beta > 0$, for all $j \leq j'$,

$$e^{-\pi\left(\frac{j}{\beta}-\alpha l\right)^2} \leq e^{-\pi\left(\frac{j'-j}{\beta}\right)^2},$$

and for all $j \geq j' + 1$,

$$e^{-\pi\left(\frac{j}{\beta}-\alpha l\right)^2} \leq e^{-\pi\left(\frac{j'-j+1}{\beta}\right)^2}.$$

Therefore,

$$C(l) \leq \sum_{j\leq j'} e^{-\pi\left(\frac{j'-j}{\beta}\right)^2} + \sum_{j\geq j'+1} e^{-\pi\left(\frac{j'-j+1}{\beta}\right)^2} = C(0) < \infty.$$

Thus the sequence $\{h_{k,l} : k, l \in \mathbb{Z}\}$ meets the requirements of Lemma 3.6.2, [Chr03], hence it is a Riesz basis sequence.

We have thus shown that $(\gamma_2, \Lambda')$ and $\{C(l) M_{\beta l} T_{\alpha(k-l)}\gamma_1 : k, l \in \mathbb{Z}\}$ are Riesz basis sequences. Furthermore, for $f = \delta_{\frac{\mathbb{Z}}{\beta}}$, the norm equivalence

$$\|\Phi_f H\|_{L^2} \asymp \|H\|_{HS},$$

holds. In conclusion, the operator family $\mathcal{H}_{F3}$ is identifiable. □

*Note*: Here $D_2(\Lambda) = \frac{1}{\sqrt{\alpha^2+\beta^2}}$ (see Table 4). We observe again that in case F3, just as in case F2, the condition $|\alpha\beta| > 1$ implies that $D_2(\Lambda) < \frac{1}{\sqrt{2}}$, *but not vice versa*!

**Proposition 4.24 (Case G1)** *Let* $\eta_0$ *be given by* (4.42), *and* $\Lambda = \begin{pmatrix} 0 & 0 \\ \beta & 0 \\ \beta & 0 \\ 0 & \alpha \end{pmatrix} \mathbb{Z}^2$ *(Table 4). If* $D_2(\Lambda) < \frac{1}{\sqrt{2}}$, *then the operator family* $\mathcal{H}_{G1} = \{H : \eta_H \in \overline{\operatorname{span}}\,(\eta_0, \Lambda)\}$ *is identifiable.*

*Proof.* The proof follows the steps outlined in Propositions 4.21, 4.22 and 4.23. The density condition on $\Lambda$ implies that $|\alpha\beta| > 1$ (see Table 4). We must check that $(\eta_0, \Lambda)$ is a Riesz basis sequence. As we have seen already this is equivalent to showing that the Gabor system $(\gamma_2, \Lambda')$ is a Riesz basis sequence where

$$\Lambda' = \begin{pmatrix} 0 & 0 \\ \frac{\beta}{\sqrt{2}} & 0 \\ \beta(1+\sqrt{2}) & 0 \\ 0 & \alpha\sqrt{2} \end{pmatrix} \begin{pmatrix} l \\ n \end{pmatrix}.$$

Observe that $(\gamma_2, \Lambda')$ is a tensor system which is a Riesz basis sequence if $|\alpha\beta| > 1$.

The rest of the proof follows the exposition of Proposition 4.23. ☐

**Proposition 4.25 (Case G3)** *Let* $\eta_0 = \chi_{[0,\frac{1}{\beta})} \otimes \chi_{[0,\frac{1}{\alpha})}$, *and* $\Lambda = \begin{pmatrix} 0 & 0 \\ \beta & 0 \\ 0 & \beta \\ \alpha & 0 \end{pmatrix} \mathbb{Z}^2$ *(see Table 4). If* $|\alpha\beta| > 1$, *then the operator family* $\mathcal{H}_{G3} = \{H : \eta_H \in \overline{\operatorname{span}}\,(\eta_0, \Lambda) \cap M^1\}$ *is identifiable.*

*Note*: The 2-density of the time-frequency lattice $\Lambda = (0, \beta m, \beta l, \alpha m) : l, m \in \mathbb{Z}^d$ is given by $\frac{1}{|\beta|\sqrt{\alpha^2+\beta^2}}$. The condition $(\alpha, \beta) : |\alpha\beta| > 1, |\beta| > 1$ implies that in case G3, if the operator family is identifiable then $D_2(\Lambda_{G3}) < \frac{1}{\sqrt{2}}$, but the converse is not true - see Proposition 4.12.

*Proof.* First, we check that $(\eta_0, \Lambda)$ is a Riesz basis sequence. However, this is clear because because the sequence $\{M_{\beta l, \alpha m}\eta_0 : m, l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2\left([0, \frac{1}{\beta}) \times [0, \frac{1}{\alpha})\right)$ and the translates $T_{0,\beta m}\eta_0$ are pairwise orthogonal (because the supports of the respective functions are pariwise disjoint due to the condition $|\beta| \geq \frac{1}{|\alpha|}$). Thus, requirement (III) is fulfilled.

The family $\mathcal{H}_{G3}$ is identifiable with $f = \delta_{\frac{\mathbb{Z}}{\beta}}$ if $\alpha\beta \geq 1$ and $\beta \geq 1$. Since $M_{\beta m} f = f$,

$$H_{l,m} f = M_{\beta l} T_{-\alpha m} H_0 T_{\alpha m} M_{\beta(m-l)} f = M_{\beta l} T_{-\alpha m} H_0 T_{\alpha m} M_{-\beta l} f.$$

The rest is solved according to the method from Theorem 3.1, [KP05]. ☐

When we change the spreading function to $\eta_0$ given by (4.42), the situation changes radically.

**Proposition 4.26 (Case G3)** *Let* $\eta_0$ *given by* (4.42), *and* $\Lambda = \begin{pmatrix} 0 & 0 \\ \beta & 0 \\ 0 & \beta \\ \alpha & 0 \end{pmatrix}$ *(see Table 4). Then the identification of the operator family* $\mathcal{H}_{G3} = \{H : \eta_H \in \overline{\operatorname{span}}\,(\eta_0, \Lambda)\}$ *is not a well-defined problem.*

*Proof.* On one hand, we must show that the Gabor system $(\gamma_2, \Lambda')$ is a Riesz basis sequence. Here

$$\Lambda' = \begin{pmatrix} 0 & 0 \\ \frac{\beta}{\sqrt{2}} & 0 \\ \beta & \beta\sqrt{2} \\ \alpha\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} m \\ l \end{pmatrix}.$$

$(\gamma_2, \Lambda')$ is a tensor Gabor system $\{M_{\beta m+\beta\sqrt{2}l}\gamma_1\}_{m,l\in\mathbb{Z}} \times \{T_{\frac{\beta m}{\sqrt{2}}} M_{\alpha\sqrt{2}l}\gamma_1\}_{m,l\in\mathbb{Z}}$. However, since the Beurling density of the set $\{\beta m + \beta\sqrt{2}l\}_{m,l\in\mathbb{Z}}$ is infinite, by Theorem 7.4.1, [CBH99], $\{M_{\beta m+\beta\sqrt{2}l}\gamma_1\}_{m,l\in\mathbb{Z}}$ can never be a frame sequence, let alone a Riesz basis sequence. Since (III) is violated, the problem is not well-posed.

□

**Proposition 4.27 (Case G4)** *Let* $\eta_0$ *be given by* (4.42), *and* $\Lambda = \begin{pmatrix} \begin{smallmatrix} 0 & 0 \\ \beta & 0 \\ 0 & \alpha \\ \beta & 0 \end{smallmatrix} \end{pmatrix} \mathbb{Z}^2$ *(Table 4). If* $\alpha, \beta$ *satisfy* $|\alpha\beta| > 1$ *and* $\frac{\alpha\sqrt{2}}{\beta} \in \mathbb{Q}$, *then the operator family* $\mathcal{H}_{G4} = \{H : \eta_H \in \overline{\text{span}}\,(\eta_0, \Lambda)\}$ *is identifiable.*

*Proof.* As in the previous examples we have to check that $(\eta_0, \Lambda_{G4})$ is a Riesz basic sequence. We use symplectic equivalence and consider instead the Gabor system $(\gamma_2, \Lambda')$. This is a Gabor system with Gaussian window and time-frequency shifts from the set $\Lambda' = \{(0, \frac{\beta}{\sqrt{2}}l, \alpha\sqrt{2}n + \beta l, \beta\sqrt{2}l) : l, n \in \mathbb{Z}\}$. $(\gamma_2, \Lambda)$ is the tensor system $\{T_{\frac{\beta}{\sqrt{2}}l} M_{\beta\sqrt{2}l}\gamma_1\}_{l\in\mathbb{Z}} \times \{M_{\alpha\sqrt{2}n+\beta l}\gamma_1\}_{l,n\in\mathbb{Z}}$. If $\frac{\alpha\sqrt{2}}{\beta} \notin \mathbb{Q}$, the set of points $\{\alpha\sqrt{2}n + \beta l : n, l \in \mathbb{Z}\}$ has an infinite upper Beurling density in $\mathbb{R}$, and $\{M_{\alpha\sqrt{2}n+\beta l}\gamma_1 : l, n \in \mathbb{Z}\}$ can never be a frame sequence (see [CBH99], especially Theorem 7.4.1, [Chr03]).

If $\{\alpha\sqrt{2}n + \beta l : l, n \in \mathbb{Z}\} \subset \{ck, k \in \mathbb{Z}\}$ for some $c \in \mathbb{R}$, then $\{M_{ck}\gamma_1 : k \in \mathbb{Z}\}$ is a Riesz basis sequence, because the function

$$\sum_{k\in\mathbb{Z}} |\gamma_1\left(\tfrac{\xi+k}{c}\right)|^2 = \sum_{k\in\mathbb{Z}} e^{-2\pi\left(\frac{\xi+k}{c}\right)^2}$$

is continuous, always positive and hence bounded above and below on $[0, 1]$ (see Theorem 7.2.3 (iii), [Chr03]).

Hence, besides the constraint $|\alpha\beta| > 1$ (in order for $\{M_{\beta l}T_{-\alpha n}H_0T_{\alpha n}f\}$ to be a Riesz basis sequence - see the previous propositions), the extra condition $\frac{\alpha\sqrt{2}}{\beta} \in \mathbb{Q}$ (equivalent to $\{\alpha\sqrt{2}n + \beta l : l, n \in \mathbb{Z}\}$ being relatively separated in $\mathbb{R}$) must be met to guarantee that $(\eta_0, \Lambda)$ is a Riesz basis sequence.

We use as identifier $f = \delta_{\beta\mathbb{Z}}$. We check that the sequence $\{H_{l,n}f\}$ is a Riesz basis for its closed linear span.

$$\begin{aligned} H_{l,n}f &= M_{\alpha n}T_{-\beta l}H_0T_{\beta l}M_{\beta l-\alpha n}\delta_{\beta\mathbb{Z}} \\ &= e^{-2\pi i(\beta l)^2}T_{\beta l}M_{\alpha n}H_0M_{\beta l-\alpha n}T_{\beta l}\delta_{\beta\mathbb{Z}} \\ &= C(n, l)T_{\beta l}M_{\alpha n}\gamma_1, \end{aligned}$$

where

$$\begin{aligned} C(n,l) &= e^{-2\pi i(\beta l)^2} H_0 M_{\beta l - \alpha n} T_{\beta l} \delta_{\beta \mathbb{Z}} \\ &= e^{-2\pi i(\beta l)^2} \sum_{j\in\mathbb{Z}} e^{-\pi(\beta j)^2} e^{2\pi i(\beta l - \alpha n)\beta j} \\ &= e^{-2\pi i(\beta l)^2} \frac{1}{\beta} \sum_{j\in\mathbb{Z}} e^{-\pi(\frac{j}{\beta}+(\beta l-\alpha n))}. \end{aligned} \tag{4.48}$$

In (4.48) we have used the Poisson summation formula for the Gaussian [Grö01]. The estimates for $\inf_{n,l\in\mathbb{Z}} C(n,l)$ and $\sup_{n,l\in\mathbb{Z}} C(n,l)$ are carried out as in Proposition 4.14. Therefore, whenever $|\alpha\beta| > 1$, $T_{\beta l} M_{\alpha n} \gamma_1$ is a Riesz basis sequence, and so is $\{H_{l,n} f\}$. □

**Proposition 4.28 (Case H4)** *Let $\eta_0$ be given by (4.42), and $\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & \alpha \\ 0 & \beta \end{pmatrix}$ (see Table 4). If $\alpha, \beta$ are such that $|\alpha| > 1$ and $\frac{\beta\sqrt{2}}{\alpha} \in \mathbb{Q}$, then the operator family $\mathcal{H}_{H4} = \{H : \eta_H \in \overline{\text{span}}\,(\eta_0, \Lambda)\}$ is identifiable.*

*Proof.* We repeat essentially the same line of reasoning as in the previous cases. First, condition (III) must be verified. By Theorem 2.2 this is equivalent to showing that the Gabor system $(\gamma_2, \Lambda')$ is a Riesz basis sequence, where

$$\Lambda' = \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ 0 & 0 \\ 0 & \alpha\sqrt{2} \\ \alpha & \beta\sqrt{2} \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix}.$$

This is a lattice containing all points $(\frac{\alpha}{\sqrt{2}}k, 0, \alpha\sqrt{2}l, \alpha k + \beta\sqrt{2}l), k, l \in \mathbb{Z}$. It is similar to the system we had in Proposition 4.24. We make some remarks about the density of the point set of the fourth coordinate of $\Lambda'$. If $\frac{\beta\sqrt{2}}{\alpha} \notin \mathbb{Q}$, then $\{\alpha k + \beta\sqrt{2}l : k, l \in \mathbb{Z}\}$ has an infinite upper Beurling density in $\mathbb{R}$, and $\{M_{\alpha k+\beta\sqrt{2}l}\gamma : k, l \in \mathbb{Z}\}$ cannot be a frame sequence at all (compare the results of [CBH99], [Chr03], especially Theorem 7.4.1). Hence, besides the constraint $|\alpha| > 1$ (which comes from the condition on the first and third coordinates of the sampling points $\frac{\alpha}{\sqrt{2}} \cdot \alpha\sqrt{2} > 1$), we have to keep in mind the extra condition that $\{\frac{\beta\sqrt{2}}{\alpha} \in \mathbb{Q}$, which guarantees that $\{\alpha k + \beta\sqrt{2}l : k, l \in \mathbb{Z}\}$ is relatively separated in $\mathbb{R}$.

If $\{\alpha k + \beta\sqrt{2}k : k, l \in \mathbb{Z}\} \subset \{cm, m \in \mathbb{Z}\}$ for some $c \in \mathbb{R}$, then $\{M_{cm}e^{-\pi x^2} : m \in \mathbb{Z}\}$ is a Riesz basis sequence, because the function

$$\sum_{m\in\mathbb{Z}} |\gamma\left(\tfrac{\xi+m}{c}\right)|^2 = \sum_{m\in\mathbb{Z}} e^{-2\pi\left(\frac{\xi+m}{c}\right)^2}$$

is continuous, always positive and hence bounded above and below on $[0,1]$ (this crietrion is given in [Chr03], Theorem 7.2.3). Therefore under the given conditions $(\eta_0, \Lambda)$ is a Riesz basis sequence and condition (III) holds.

We select as identifier $f = \delta_{\frac{1}{\alpha}\mathbb{Z}}$. We must show that $\{H_{k,l}f\}$ is a Riesz basis sequence too. We obtain just like in case H1:

$$H_{k,l}f = e^{2\pi i \alpha l \beta l} T_{\alpha k - \beta l} M_{\alpha l} H_0 T_{\beta l} M_{-\alpha l} f = T_{\alpha k - \beta l} M_{\alpha l} H_0 T_{\beta l} \delta_{\frac{1}{\alpha}\mathbb{Z}}.$$

and

$$\begin{aligned} H_0 T_{\beta l} \delta_{\frac{1}{\alpha}\mathbb{Z}} &= \int \kappa_0(x,t) \sum_{j\in\mathbb{Z}} \delta_{\frac{1}{\alpha}j}(t-\beta l)dt \\ &= \sum_{j\in\mathbb{Z}} \kappa_0(x, \frac{j}{\alpha} - \beta l) \\ &= e^{-\pi x^2} \sum_{j\in\mathbb{Z}} e^{-\pi(\frac{j}{\alpha}-\beta l)^2} \end{aligned} \tag{4.49}$$

Next we must show that the sequence $\{H_{k,l} : k,l \in \mathbb{Z}\}$ associated to $f = \delta_{\alpha\mathbb{Z}}$ is a Riesz sequence. Its elements are

$$H_{k,l}f = C(l) T_{\alpha k - \beta l} M_{\alpha l} \gamma_1 : k,l \in \mathbb{Z}, \tag{4.50}$$

where we denote

$$C(l) = e^{2\pi i \alpha l \beta l} \sum_{j\in\mathbb{Z}} e^{-\pi\left(\frac{j}{\alpha}-\beta l\right)^2}.$$

The rest of the proof proceeds along the line of reasoning given in Proposition 4.26 since the sequence $\{T_{\alpha k-\beta l} M_{\alpha l}\gamma_1 : k,l \in \mathbb{Z}\} = (\gamma_1, \left(\begin{smallmatrix} \alpha & -\beta \\ 0 & \alpha \end{smallmatrix}\right) \mathbb{Z}^2)$ is a Riesz sequence if $|\alpha| > 1$ [SW92], [Hei07] (Theorem 14). □

*Note:* The 2-density of the time-frequency lattice in this case is $\frac{1}{|\alpha|\sqrt{\alpha^2+\beta^2}}$. Our assumptions show that $\mathcal{H}_{H1}$ being identifiable implies $D_2(\Lambda) < 1$, but not vice versa - see Proposition 4.12. The condition $\frac{\beta\sqrt{2}}{\alpha} \in \mathbb{Q}$, which makes the problem well-posed, is not even quantified in $D_2(\Lambda)$.

**Proposition 4.29 (Case H5)** *Let $\eta_0$ be given by (4.42), and $\Lambda = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)$ (Table 4). If $\alpha, \beta$ are such that $|\alpha| > 1, |\alpha\beta| > 1, |(\alpha-\beta)\alpha| > 1$, then the operator family $\mathcal{H}_{H5} = \{H : \eta_H \in \overline{\operatorname{span}}\,(\eta_0, \Lambda)\}$ is identifiable.*

*Proof.* The first step of the proof is to verify the Riesz basis sequence condition (III) for $(\eta_0, \Lambda)$. By symplectic equivalence (Theorem 2.2) this is equivalent to checking that the Gabor system $(\gamma_2, \Lambda')$ is a Riesz basis sequence, where

$$\Lambda' = \begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ 0 & 0 \\ 0 & \alpha\sqrt{2} \\ \alpha+\beta\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} k \\ n \end{pmatrix}.$$

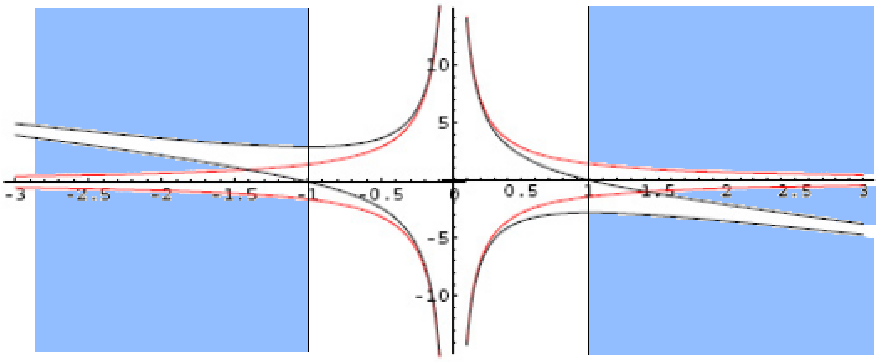


Figure 2: The set $(\alpha, \beta)$ fulfilling the conditions in case H5 lies in the blue region.

This is a Gabor system with a time-frequency lattice, containing the points $(\frac{\alpha}{\sqrt{2}}k, 0, \alpha\sqrt{2}n, (\alpha+\beta\sqrt{2})k)$, $k, n \in \mathbb{Z}$, which is similar to the systems we have encountered so far. If $|\alpha| > 1$, $(\gamma_2, \Lambda')$ is a Riesz basis sequence, and consequently (III) holds.

We choose for the identifier $f = \delta_{\frac{1}{\alpha}\mathbb{Z}}$. Second, we must show that $\{H_{k,n}f\}$ is a Riesz basis sequence.

$$H_{k,n}f = T_{(\alpha-\beta)k}M_{\alpha n}H_0 T_{\beta k}M_{-\alpha n}f = C(k)T_{(\alpha-\beta)k}M_{\alpha n}\gamma_1$$

where we denote

$$C(k) = \frac{1}{\alpha}\sum_{j\in\mathbb{Z}} e^{-\pi\left(\frac{j}{\alpha}-\beta k\right)^2}.$$

From here on we pursue the same path as in Proposition 4.26. When $|(\alpha-\beta)\alpha| > 1$, the sequence $\{H_{k,n}f\}$ is a Riesz basis for its closed linear span. This shows the claim. □

*Note*: The 2-density of the time-frequency lattice in the case H5, $D_2(\Lambda) = \frac{1}{|\alpha|\sqrt{\alpha^2+\beta^2}}$. Our assumptions show that $\mathcal{H}_{H5}$ identifiable implies $D_2(\Lambda) < \frac{1}{\sqrt{2}}$, *but not vice versa* - see Proposition 4.12 and Proposition 4.28. The set of values $\alpha, \beta$ which satisfy the conditions of Proposition 4.29 are listed in Figure 2.

**Proposition 4.30 (Case H6)** *Let $\eta_0$ be given by (4.42), and $\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \\ \alpha & 0 \\ 0 & \beta \end{pmatrix}$ (Table 4). If $\alpha, \beta$ are such that $\frac{\beta\sqrt{2}}{\alpha} \in \mathbb{Q}, |\alpha\beta| > 1$, then the operator family $\mathcal{H}_{H6} = \{H : \eta_H \in \overline{\text{span}}\,(\eta_0, \Lambda)\}$ is identifiable.*

*Proof.* This is similar to Proposition 4.28. We note that in this particular case $(\gamma_2, \Lambda') = \{T_{\frac{\alpha}{\sqrt{2}}k}M_{\alpha\sqrt{2}k}\gamma_1\} \times \{M_{\alpha k+\beta\sqrt{2}l}\}$ is a Riesz sequence if $\frac{\beta\sqrt{2}}{\alpha} \in \mathbb{Q}$. The rest is essentially identical to case H4. □

## 4.7 Other spreading functions

In contrast to the previous section where $\eta_0$ was fixed, and $\Lambda$ was varying, in the following examples we shall vary both $\eta_0$ (requirement II) and $\Lambda$ (requirement I) to produce identifiable families $\mathcal{H}_\Lambda$. We discuss briefly some special $\eta_0$, which are distributions, so that the prototype operator $H_0$ is non-Hilbert-Schmidt, but of a special type (convolution, multiplication, etc.). The forms of $H_0 f$ are listed in Table 1. For the purpose of obtaining expansions for $\eta_H$ in terms of a Riesz basis $(\eta_0, \Lambda)$ (requirement III), we pose various initial conditions on $\eta_0$.

We start with the simplest case $\eta_0(t,\nu) = \delta_{0,0}(t,\nu)$. In this case, the prototype operator $H_0$ is the identity. The family of operators generated by Gabor Riesz basis sequence expansions of such $H_0$ is not Hilbert-Schmidt.

A brief check of the different cases from Tables 2 and 3 we see that in cases B2-B6, D5, E5, F6, G4-G6, H4-H6, the sequence $(\eta_0, \Lambda)$ can never be a Riesz basis sequence. So requirement (III) is violated. We summarize our conclusions about the remaining cases B1, D1, D3, E1, E3, F1, F2 in

**Proposition 4.31** *Let $\eta_0(t,\nu) = \delta_{0,0}(t,\nu)$. The set of spreading functions arising from cases B1, D1, D3, E1, E3, F1, F2 from Table 2 and 3 contains only the Gabor sysnthesis operator $D_{f,\alpha,\beta}$ and is identifiable with any $f \in L^2$ such that $(f, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a Riesz basis sequence.*

*Proof.* The fact that the operator is Gabor synthesis results from the substitution $H_0 = Id$ into the formulae of Table 2 and Table 3, namely.

$$\Phi_f : \mathbf{c} \mapsto D_{f,\alpha,\beta}\mathbf{c}, \quad \mathbf{c} \in \ell^2(\mathbb{Z}^2)$$

The result follows. □

A second special case of spreading function is $\eta_0(t,\nu) = p(t)\delta_0(\nu)$, whose associated operator $H_0$ is a convolution operator,

$$H_0 : f \mapsto p * f, \quad f \in L^2(\mathbb{R}).$$

We consider the various time-frequency index sets $\Lambda$. Some formulas for a representative operator $H \in \mathcal{H}_\Lambda$ are listed in column II of Table 1. We note that in cases B2, B3, B5, D2, D4, E3, E5, F2, F4, G1, H2, H6 requirement (III) is not fulfilled, so these can be excluded from our consideration.

For $H_0$ being a convolution $H_0 : f \mapsto p * f$, we will consider lattices $\Lambda$ listed as B1, D1, D3, H1, F1. Note that $p$ and conditions on $\Lambda$ will be different in each case to ensure that $\mathcal{H}_\Lambda$ is identifiable.

*Note*: We must ensure in each case that $\kappa_H, f$ belong to a pair of dual spaces so that the integral (2.11) for $Hf$ is well-defined. The kernel $\kappa_H(x,t) = p(t) \in L^2(\mathbb{R})$, and the identifier $f$ must also belong to $L^2(\mathbb{R})$, so that $Hf$ is well-defined.

Cases B1, D1, D3 are quite similar to each other.

| Case | Prototype operator | Operator representation |
| --- | --- | --- |
| B1 | $f \mapsto T_{\alpha k}(p * M_{\gamma m} f)$ | $f \mapsto \sum_{k,m} c_{k,m} T_{\alpha k}(p * M_{\gamma m} f)$ |
| B3 | $f \mapsto T_{\alpha k}(p * f)$ | $f \mapsto \sum_{k,n} c_{k,n} T_{\alpha k}(p * f)$ |
| B5 | $f \mapsto p * M_{\gamma m} f$ | $f \mapsto \sum_{m,n} c_{m,n} p * M_{\gamma m} f$ |
| D1 | $f \mapsto p * T_{\alpha k} M_{\beta m} f$ | $f \mapsto \sum_{k,m} c_{k,m} p * T_{\alpha k} M_{\beta m} f$ |
| D2 | $f \mapsto T_{\alpha k} p * M_{\beta k} f$ | $f \mapsto \sum_{k,n} c_{k,n} T_{\alpha k} p * M_{\beta k} f$ |
| D3 | $f \mapsto T_{\alpha k} p * M_{\beta n} f$ | $f \mapsto \sum_{k,n} c_{k,n} T_{\alpha k} p * M_{\beta n} f$ |
| E1 | $f \mapsto M_{\beta l} T_{\alpha k} p * f$ | $f \mapsto \sum_{k,l} c_{k,l} M_{\beta l} T_{\alpha k} p * f$ |
| E3 | $f \mapsto T_{\alpha k} M_{\beta k} p * M_{\beta(m-k)} f$ | $f \mapsto \sum_{k,m} c_{k,m} T_{\alpha k} M_{\beta k} p * M_{\beta(m-k)} f$ |
| F1 | $f \mapsto M_{\beta l} p * T_{\alpha k} f$ | $f \mapsto \sum_{k,l} c_{k,l} M_{\beta l} p * T_{\alpha k} f$ |
| F2 | $f \mapsto T_{\alpha k} M_{\beta m} p * M_{\beta(m-k)} f$ | $f \mapsto \sum_{k,m} c_{k,m} T_{\alpha k} M_{\beta m} p * M_{\beta(m-k)} f$ |
| F3 | $f \mapsto M_{\beta l} T_{\alpha k} p * M_{\beta(k-l)} f$ | $f \mapsto \sum_{k,l} c_{k,l} M_{\beta l} T_{\alpha k} p * M_{\beta(k-l)} f$ |
| G1 | $f \mapsto M_{\beta l} p * f$ | $f \mapsto \sum_{l,n} c_{l,n} M_{\beta l} p * f$ |
| G2 | $f \mapsto M_{\beta l} p * M_{\beta(m-l)} f$ | $f \mapsto \sum_{m,l} c_{m,l} M_{\beta l} p * M_{\beta(m-l)} f$ |
| G3 | $f \mapsto M_{\beta l} p * M_{\beta(m-l)} f$ | $f \mapsto \sum_{m,l} c_{m,l} M_{\beta l} p * M_{\beta(m-l)} f$ |
| H1 | $f \mapsto T_{\alpha k} M_{\beta l} p * M_{-\beta l} f$ | $f \mapsto \sum_{k,l} c_{k,l} T_{\alpha k} M_{\beta l} p * M_{-\beta l} f$ |
| H2 | $f \mapsto T_{\alpha k} M_{\beta k} p * M_{-\beta k} f$ | $f \mapsto \sum_{k,l} c_{k,l} T_{\alpha k} M_{\beta k} p * M_{-\beta k} f$ |

Table 5: Different operator classes induced by $\eta_0 = p(t)\delta_0(\nu)$.

**Proposition 4.32 (Case B1)** *Let $\alpha\gamma \geq 1$, $p \in L^2(\mathbb{R})$ such that* $\operatorname{supp} \widehat{p} \subseteq [0, \gamma)$ *and $\{T_{\alpha k} p : k \in \mathbb{Z}\}$ is a Riesz basis sequence. Then the operator class*

$$\mathcal{H} = \{H : \eta_H \in \overline{\operatorname{span}} \{T_{\alpha k, \gamma m} \eta_0\}\}$$

*is identifiable with identifier $f = \widehat{\chi}_{[0,\gamma)}$.*

*Proof.* For this set-up, a typical representative of $\mathcal{H}$ is

$$H : f \mapsto \sum c_{k,m} (T_{\alpha k} p * M_{\gamma m} f), \quad \mathbf{c} \in \ell^2(\mathbb{Z}^2).$$

We denote for the sake of shortness $H_{k,m} f := T_{\alpha k} p * M_{\gamma m} f$. As usual, the proof consists of 2 steps: first, verifying condition (III) for $\eta_0$ and second, showing that for this choice of identifier $f$, $\{H_{k,m} f\}$ is a Riesz basis sequence.

First, we justify the condition $\alpha\gamma \geq 1$ in the light of condition (III). A well-known condition for Riesz basis sequences of translates (cited, for example, in [Chr03]) states that it is sufficient to have

$$\sum_{k \in \mathbb{Z}} \left| \widehat{p}\left(\xi + \tfrac{k}{\alpha}\right) \right|^2 \asymp 1 \quad \text{almost everywhere on } [0, \tfrac{1}{\alpha})$$

If $\operatorname{supp} \widehat{p} \subseteq [0, \gamma)$, whenever $\alpha\gamma < 1$, the above norm equivalence does not hold. Hence, $\alpha\gamma \geq 1$ is a necessary requirement.

Second, we show that for $f = \widehat{\chi}_{[0,\gamma)}$ the function sequence $\{H_{k,m}f : m, k \in \mathbb{Z}\}$ is a Riesz basis sequence. The Riesz basis property of a sequence $\{e_j\}$ carries over to the sequence $\{\widehat{e}_j\}$ because the Fourier transform is a unitary map [Chr03]. So we work in the Fourier domain. We take the inner product

$$\begin{aligned}\langle H_{k_1,m_1}f, &H_{k_2,m_2}f\rangle = \\ &= \langle T_{\alpha k_1}p * M_{\gamma m_1}f, T_{\alpha k_2}p * M_{\gamma m_2}f\rangle \\ &= \langle M_{-\alpha k_1}\widehat{p} \cdot T_{\gamma m_1}\widehat{f}, M_{-\alpha k_2}\widehat{p} \cdot T_{\gamma m_2}\widehat{f}\rangle \\ &= \int M_{-\alpha k_1}\widehat{p}(\xi)\overline{M_{-\alpha k_2}\widehat{p}(\xi)}\widehat{f}(\xi - \gamma m_1)\overline{\widehat{f}(\xi - \gamma m_2)}\, d\xi \\ &= \delta_0(m_1 - m_2)\int_0^\gamma M_{-\alpha k_1}\widehat{p}(\xi)\overline{M_{-\alpha k_2}\widehat{p}(\xi)}\, d\xi \end{aligned} \tag{4.51}$$

The last equality holds because $\widehat{f} = \chi_{[0,\gamma)}$, which implies that

$$\widehat{f}(\xi - \gamma m_1)\overline{\widehat{f}(\xi - \gamma m_2)} = 0$$

when $m_1 \neq m_2$. Hence we obtain that

$$\widehat{f}(\xi - \gamma m_1)\overline{\widehat{f}(\xi - \gamma m_2)} = \delta_0(m_1 - m_2)\chi_{[0,\gamma)}(\xi)$$

so we can pull that term outside the integral.

Due to the assumption $\operatorname{supp}\widehat{p} \subseteq [0, \gamma)$, the last line of (4.51) is simply

$$\delta_0(m_1 - m_2)\langle M_{-\alpha k_1}\widehat{p}, M_{-\alpha k_2}\widehat{p}\rangle,$$

so we can take inverse Fourier transform of both sides of the inner product and obtain the equality

$$\langle H_{k_1,m_1}f, H_{k_2,m_2}f\rangle = \delta_0(m_1 - m_2)\langle T_{\alpha k_1}p, T_{\alpha k_2}p\rangle.$$

Next we use the criterion for Riesz bases from (2.1). We take a finite sequence $\mathbf{c} = \{c_{k,m} : k, m \in \mathbb{Z}\}$ and use the above equality to compute

$$\|\sum c_{k,m}H_{k,m}f\|_{L^2}^2 = \sum_m \|\sum_k c_{k,m}T_{\alpha k}p\|_{L^2}^2 \asymp \|\mathbf{c}\|_{\ell^2}$$

because $\{T_{\alpha k}p\}$ is a Riesz basis sequence. Hence $\{H_{k,m}f\}$ is a Riesz basis sequence.

Thus, the mapping $\Phi_f : \mathbf{c} \mapsto \sum c_{k,m}H_{k,m}f$ is bounded and has a bounded inverse, and $\mathcal{H}$ is identifiable under the assumptions of Proposition 4.32. □

This method can be applied to the operator families in cases D1 and D3 with small adjustments of the initial conditions.

**Proposition 4.33 (case D1)** *Let $|\alpha\beta| \geq 1$, $p \in L^2(\mathbb{R})$ such that $\operatorname{supp}\widehat{p} \subseteq [0, \beta)$ and $\{T_{\alpha k}p : k \in \mathbb{Z}\}$ is a Riesz basis sequence. Then the operator class*

$$\mathcal{H} = \{H : \eta_H \in \overline{\operatorname{span}}\{T_{\alpha k,\beta m}M_{0,\alpha k}\eta_0\}\},$$

*is identifiable with identifier $f = \widehat{\chi}_{[0,\beta)}$.*

*Proof.* A typical representative of $\mathcal{H}$ is an operator of the form

$$H : f \mapsto \sum c_{k,m} p * T_{\alpha k} M_{\beta m} f, \quad \mathbf{c} \in \ell^2(\mathbb{Z}^2).$$

We note that the condition $|\alpha\beta| \geq 1$ is necessary to ensure the existence of a function $p$ satisfying the conditions

1. $\operatorname{supp} \widehat{p} \subseteq [0, \beta)$;
2. $\{T_{\alpha k} M_{\alpha k} p : k \in \mathbb{Z}\}$ is a Riesz basis sequence.

The rest of the proof follows that of Proposition 4.32 since

$$H_{k,m} f = p * T_{\alpha k} M_{\beta m} f = T_{\alpha k} p * M_{\beta m} f.$$

□

**Proposition 4.34 (Case D3)** *Let* $|\alpha\beta| \geq 1$, $p \in L^2(\mathbb{R})$ *such that* $\operatorname{supp} \widehat{p} \subseteq [0, \beta)$, *and* $\{T_{\alpha k} p : k \in \mathbb{Z}\}$ *is a Riesz basis sequence. Then the operator family*

$$\mathcal{H} = \{H : \eta_H \in \overline{\operatorname{span}} \{T_{\alpha k, \beta n} M_{\alpha n, 0} \eta_0 : k, n \in \mathbb{Z}\}\}$$

*is identifiable with identifier* $f = \widehat{\chi}_{[0,\beta)}$.

*Proof.* This example is similar to Proposition 4.32 and 4.33. A typical operator from this class maps

$$H : f \mapsto \sum c_{k,n} (T_{\alpha k} p * M_{\beta n} f), \quad \mathbf{c} \in \ell^2(\mathbb{Z}^2).$$

The rest follows the steps of Proposition 4.32 and 4.33. □

**Proposition 4.35 (Case H1)** *Let* $|\alpha\beta| > 1$ *and* $p \in \mathcal{S}(\mathbb{R})$ *be such that* $(p, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ *is a Gabor Riesz basis. Then the operator family*

$$\mathcal{H} = \{H : \eta_H \in \overline{\operatorname{span}} \{T_{\alpha k, 0} M_{\alpha l, \beta k} \eta_0 : k, l \in \mathbb{Z}\}\}$$

*is identifiable with* $f = \delta_0$.

**Note:** This identifier from $\mathcal{S}'(\mathbb{R})$ is admissible because $\kappa_H(x,t) = p(t) \in \mathcal{S}(\mathbb{R})$.

*Proof.* The initial condition on $p$ coming from (III) is that $\{T_{\alpha k} p : k \in \mathbb{Z}\}$ must be a Riesz basis sequence. This is however satisfied automatically because this is a subsequence of $(p, \alpha\mathbb{Z} \times \beta\mathbb{Z})$.

A typical representative of this operator family is

$$H : f \mapsto \sum_{k,l} c_{k,l} T_{\alpha k} M_{\beta l} p * M_{-\beta l} f, \quad c \in \ell^2(\mathbb{Z}^2)$$

If $f = \delta_0$, then the evaluation map $\Phi_f$ is equivalent to

$$\Phi_f : \mathbf{c} \mapsto D_{p,\alpha,\beta} \mathbf{c},$$

which represents the synthesis operator of a Gabor system $(p, \alpha\mathbb{Z} \times \beta\mathbb{Z})$. According to the assumption that $(p, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a Gabor Riesz basis, we obtain that $\Phi_f$ is bounded and has a bounded inverse. □

The final example for this choice of $\eta_0$ considers the lattice from case F1.

**Proposition 4.36 (Case F1)** *Let $p \in \mathcal{S}(\mathbb{R})$ such that $(p, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a Riesz basis sequence. Then the family of operators*

$$\mathcal{H} = \{H : \eta_H \in \overline{\text{span}}\,\{T_{\alpha k,\beta l} M_{\beta l,\alpha k} \eta_0\}\,\}$$

*is identifiable with $f = \delta_0$.*

*Note*: This identifier from $\mathcal{S}'(\mathbb{R})$ is admissible because $\kappa_H(x,t) = p(t) \in \mathcal{S}(\mathbb{R})$. We must therefore have in addition the condition $\alpha\beta > 1$.

*Proof.* The initial condition on $p$ coming from (III) is the sequence $\{T_{\alpha k} M_{\beta l} p : k, l \in \mathbb{Z}\}$ to be a Riesz basis sequence.

A typical representative of $\mathcal{H}$ is

$$H : f \mapsto \sum_{k,l} c_{k,l} M_{\beta l}(p * T_{\alpha k} f), \quad c \in \ell^2(\mathbb{Z}^2).$$

For the identifier $f = \delta_0$ we have the following

$$(p * T_{\alpha k}\delta_0)(x) = \int p(x-t)\delta_0(t - \alpha k)\,dt = p(x - \alpha k) = T_{\alpha k} p(x)$$

Therefore, the evaluation map is

$$\Phi_{\delta_0} : c \mapsto D_{p,\alpha,\beta} c,$$

which is bounded and invertible if $(p, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a Riesz basis sequence in $L^2(\mathbb{R})$. □

A third special case of generator is $\eta_0 = \delta_0(t) q(\nu)$. It corresponds to a multiplication operator

$$H_0 : f \mapsto f \cdot \overline{\widehat{q}},$$

with operator kernel $k_0(x,t) = \overline{\widehat{q}(x)}\delta_0(t)$. Hence we should always choose an identifier in $\mathcal{S}(\mathbb{R})$ in order to have the integral (2.11) for $Hf$ well-defined.

An inspection of the conditions on the parameters of $\Lambda$ from Table 2 and 3 imposed by the (III) shows us that for lattices in cases B2-4, B6, D2, E2, G1, G3 and H2 the problem is not well-defined. An inspection of Table 6 shows that the operator class $\mathcal{H}_\Lambda$ in cases H3 are not identifiable.

We consider the time-frequency index sets $\Lambda$ from case E3 and E1 (Table 2), which being again very similar to each other, we will combine them into a single proposition.

**Proposition 4.37** *Assume* $\text{supp}\,\widehat{q} \subset \left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right)$ *and that $\{T_{\beta k} q : k \in \mathbb{Z}\}$ is a Riesz basis sequence (i.e. $|\alpha\beta| > 1$). Then the operator classes*

1. $\mathcal{H}_{E3} = \{H : \eta_H \in \overline{\text{span}}\,\{T_{\alpha k,\beta m} M_{\beta k,0} \eta_0\}\,\}$
2. $\mathcal{H}_{E1} = \{H : \eta_H \in \overline{\text{span}}\,\{T_{\alpha k,\beta l} M_{\beta l,0} \eta_0\}\,\}$

| Case | Prototype operator | Operator representation |
|---|---|---|
| B1 | $f \mapsto T_{\alpha k} M_{\gamma m}(f \cdot \overline{\widehat{q}})$ | $f \mapsto D_{f \cdot \overline{\widehat{q}}, \alpha, \gamma} \mathbf{c}$ |
| B4 | $f \mapsto M_{\gamma m} f \cdot \overline{\widehat{q}}$ | $f \mapsto \sum_{m,l} c_{m,l} M_{\gamma m} f \cdot \overline{\widehat{q}}$ |
| B6 | $f \mapsto T_{\alpha k}(f \cdot \overline{\widehat{q}})$ | $f \mapsto \sum_{k,l} c_{k,l} T_{\alpha k}(f \cdot \overline{\widehat{q}})$ |
| D1 | $f \mapsto T_{\alpha k} M_{\beta m} f \cdot \overline{\widehat{q}}$ | $f \mapsto \sum_{k,m} c_{k,m} T_{\alpha k} M_{\beta m} f \cdot \overline{\widehat{q}}$ |
| D3 | $f \mapsto \overline{T_{\alpha(k-n)} \widehat{q}} \cdot T_{\alpha k} M_{\beta n} f$ | $f \mapsto \sum_{k,n} c_{k,n} \overline{T_{\alpha(k-n)} \widehat{q}} \cdot T_{\alpha k} M_{\beta n} f$ |
| E1 | $f \mapsto M_{\beta l} T_{\alpha k}(\overline{\widehat{q}} \cdot f)$ | $f \mapsto \sum_{k,l} c_{k,l} M_{\beta l} T_{\alpha k}(\overline{\widehat{q}} \cdot f)$ |
| E2 | $f \mapsto \overline{T_{\alpha k} \widehat{q}} \cdot T_{\alpha k} M_{\beta k} f$ | $f \mapsto \sum_{k,l} c_{k,l} \overline{T_{\alpha k} \widehat{q}} \cdot T_{\alpha k} M_{\beta k} f$ |
| E3 | $f \mapsto \overline{T_{\alpha k} \widehat{q}} \cdot T_{\alpha k} M_{\beta m} f$ | $f \mapsto \sum_{k,m} c_{k,m} \overline{T_{\alpha k} \widehat{q}} \cdot T_{\alpha k} M_{\beta m} f$ |
| F1 | $f \mapsto \overline{M_{\beta l} \widehat{q}} \cdot T_{\alpha k} f$ | $f \mapsto \sum_{k,l} c_{k,l} \overline{M_{\beta l} \widehat{q}} \cdot T_{\alpha k} f$ |
| F2 | $f \mapsto T_{\alpha k} M_{\beta m} f \cdot \overline{T_{\alpha(m-k)} \widehat{q}}$ | $f \mapsto \sum_{k,m} c_{k,m} T_{\alpha k} M_{\beta m} f \cdot \overline{T_{\alpha(m-k)} \widehat{q}}$ |
| F3 | $f \mapsto T_{\alpha k} M_{\beta l} f \cdot \overline{T_{\alpha(k-n)} \widehat{q}}$ | $f \mapsto \sum_{k,l} c_{k,l} T_{\alpha k} M_{\beta l} f \cdot \overline{T_{\alpha(k-n)} \widehat{q}}$ |
| G2 | $f \mapsto \overline{T_{-\alpha l} \widehat{q}} \cdot M_{\beta m} f$ | $f \mapsto \sum_{l,m} c_{l,m} \overline{T_{-\alpha l} \widehat{q}} \cdot M_{\beta m} f$ |
| G3 | $f \mapsto \overline{T_{-\alpha m} \widehat{q}} \cdot M_{\beta m} f$ | $f \mapsto \sum_{l,m} c_{l,m} \overline{T_{-\alpha m} \widehat{q}} \cdot M_{\beta m} f$ |
| H1 | $f \mapsto T_{\alpha k} f \cdot \overline{T_{\alpha(k-l)} \widehat{q}}$ | $f \mapsto \sum_{k,l} c_{k,l} T_{\alpha k} f \cdot \overline{T_{\alpha(k-l)} \widehat{q}}$ |
| H3 | $f \mapsto T_{-\alpha k} f \cdot \widehat{q}$ | $f \mapsto \sum_{k,l} c_{k,l} T_{-\alpha k} f \cdot \widehat{q}$ |

Table 6: Different operator classes induced by $\eta_0 = \delta_0(t) q(\nu)$.

*are identifiable with any* $f \in \mathcal{S}(\mathbb{R})$ *such that* $f|_{[-\frac{\alpha}{2}, \frac{\alpha}{2})} = 1$.

*Proof.* We consider first Case E3. The condition $|\alpha\beta| > 1$ is required to satisfy the requirement for Riesz basis sequences of translates [Chr03]. Let us assume that $\operatorname{supp} \widehat{q} \subset [-\frac{\alpha}{2}, \frac{\alpha}{2})$. A typical representative of $\mathcal{H}$ is

$$H : f \mapsto \sum_{k,m} c_{k,m} \overline{T_{\alpha k} \widehat{q}} \cdot T_{\alpha k} M_{\beta m} f.$$

We pick as identifier $f \in S(\mathbb{R})$ such that $f|_{[-\frac{\alpha}{2}, \frac{\alpha}{2})} = 1$. Then for a finitely supported sequence $\mathbf{c} = \{c_{k,m}\}$ we have

$$\begin{aligned}
&\|\sum_{k,m} c_{k,m} T_{\alpha k}\widehat{q} \cdot T_{\alpha k} M_{\beta m} f\|_2^2 = \\
&\qquad = \sum_{k,m}\sum_{k',m'} c_{k,m}\overline{c_{k',m'}} \langle \overline{T_{\alpha k}\widehat{q}} \cdot T_{\alpha k} M_{\beta m} f, \overline{T_{\alpha k'}\widehat{q}} \cdot T_{\alpha k'} M_{\beta m'} f \rangle \\
&\qquad = \sum_{k,m}\sum_{k',m'} c_{k,m}\overline{c_{k',m'}} \int \overline{T_{\alpha k}\widehat{q}(t)} T_{\alpha k} M_{\beta m} f(t) T_{\alpha k'}\widehat{q}(t) \overline{T_{\alpha k'} M_{\beta m'} f(t)} dt \\
&\qquad = \sum_{k,m}\sum_{k',m'} c_{k,m}\overline{c_{k',m'}} \delta_0(k-k') \int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} |\widehat{q}(t)|^2 M_{\beta m} f(t) \overline{M_{\beta m'} f(t)} dt \\
&\qquad = \sum_{k}\sum_{m,m'} c_{k,m}\overline{c_{k,m'}} \int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} M_{\beta m}\widehat{q}(t) \overline{M_{\beta m'}\widehat{q}(t)} dt \\
&\qquad \asymp \|\mathbf{c}\|_{\ell^2}
\end{aligned}$$

The norm equivalence in the last line is due to $\{M_{\beta m}\widehat{q} : m \in \mathbb{Z}\}$ being a Riesz basis sequence (2.1). The latter is the Fourier transform of a subsequence of the Riesz basis sequence $\{T_{\beta m} q\}$ (follows from the initial condition on $q$ in the assumption of the Proposition). Hence the evaluation map $\Phi_f$ is bounded and has a bounded inverse.

A similar proof holds for case E1. □

A fifth special case is $\eta_0 = V_{g_1} g_2(t,\nu)$. The operator corresponding to $\eta_0$ is

$$H_0 f = g_2 \langle f, g_1 \rangle \tag{4.52}$$

We see that the operator classes resulting in cases B1, D1 and F for such $\eta_0$ are never identifiable.

**Proposition 4.38** *Let $\eta_0 = V_{g_1} g_2(t,\nu)$ for some $g_1, g_2 \in M^1(\mathbb{R}^d)$. Then the operator class given by*

1. $\mathcal{H}_{B1} = \{H : \eta_H \in \overline{\operatorname{span}}\,\{T_{\alpha k,\gamma m}\eta_0\}\,\}$;
2. $\mathcal{H}_{D1} = \{H : \eta_H \in \overline{\operatorname{span}}\,\{T_{\alpha k,\beta m} M_{\alpha k,0}\eta_0\}\,\}$;
3. $\mathcal{H}_{F1} = \{H : \eta_H \in \overline{\operatorname{span}}\,\{T_{\alpha k,\beta l} M_{\alpha k,\beta l}\eta_0\}\,\}$.

*is not identifiable.*

*Proof.*

1. The underlying lattice is listed as case B1 from Figure 2. A typical representative of the associated operator family is

$$Hf = \sum c_{k,m} \langle f, M_{\gamma m} g_1 \rangle T_{\alpha k} g_2.$$

which can be rewritten as

$$Hf = \sum_k \left( \sum_m c_{k,m} \langle f, M_{\gamma m} g_1 \rangle \right) T_{\alpha k} g_2. \quad (4.53)$$

The expression in brackets in (4.53) is the convolution of the sequences $\mathbf{c}_k(m)$ and $\{\langle f, M_{\gamma m} g_1 \rangle\}_m$. This representation shows immediately that the coefficients $c_{k,m}$ are not recoverable, so this case is not identifiable.

2. The underlying lattice is from case D1 from Figure 2. The action of the operator $H$ on $f$ we shall rewrite as follows

$$Hf = g_2 \langle \sum c_{k,m} T_{\alpha k} M_{\beta m} f, g_1 \rangle = \langle \mathbf{c}, C_{g_1,\alpha,\beta} f \rangle_{\ell^2} g_2 \quad (4.54)$$

because $\langle T_{\alpha k} M_{\beta m} f, g_1 \rangle$ is basically the action of the analysis operator $C_{g_1,\alpha,\beta}$ on $f$. It is easy to see that $\mathbf{c}$, hence $H$ is never recoverable from the value on the right-hand side of (4.54).

3. This lattice is from case F1 from Figure 3. A typical representative of the operator family is

$$Hf = \sum c_{k,l} \langle f, T_{-\alpha k} g_1 \rangle M_{\beta l} g_2,$$

and rewrite it as

$$Hf = \sum_l \big( \sum_k \langle f, T_{-\alpha k} g_1 \rangle \big) M_{\beta l} g_2. \quad (4.55)$$

The expression on the right-hand side of (4.55) is similar to (4.54). Similar to 2. we conclude that the operator class is not identifiable.

□

**Note:** Overall for such $\eta_0$ it is more difficult to have initial conditions as required by (III) for $g_1, g_2$.

As a conclusion to Section 4 we summarize the results as

**Theorem 4.39** *There exists no universal constant $c$ such that the operator family $\mathcal{H}_\Lambda$ is identifiable if $D_2(\Lambda) < c$.*

The examples cited show that identification of an operator class can not be dependent on a single parameter such as 2-density. For different time-frequency index sets $\Lambda$ there exists different constants. Furthermore, there exist families (Proposition 4.12) where identification can not be expressed in terms of a 2-density at all. For other families, extra conditions must be imposed to make the problem well-posed (Proposition 4.15)

# 5 Localization, HAPs and Gabor molecules

Section 4 showed that the answer to the question: 'Under which conditions is identification of a given family $\mathcal{H}_\Lambda$ possible?' depends very much on the interplay between the criteria listed as (I)-(V) in Section 4. Here we approach the identification problem from a different angle. We shall use the assumptions

(I) $\Lambda = A\mathbb{Z}^2 \in \mathbb{R}^4$, $A$ is a $4 \times 2$-matrix;

(II) $\eta_0 \in M^1_v(\mathbb{R}^2)$, $v$ a polynomial weight of degree strictly greater than 2;

(III) $(\eta_0, \Lambda)$ is a Riesz basis sequence.

(IV) There exists $c > 0$, such that $D_2(\Lambda) < c \Longrightarrow \mathcal{H}_\Lambda$ is identifiable.

Our goal in Section 5 is to explore the admissible range of constants $c$ for (IV) in order for the identification problem to be *well-posed* in the most general sense. In other words, we search for the *broadest range* of constants $C$ such that the requirement (III) holds.

Let $\eta_0 \in M^1_s(\mathbb{R}^2)$ with associated Hilbert-Schmidt operator $H_0$. Let

$$\Lambda = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} \mathbb{Z}^2$$

Assume (III), that is,

$$\{T_{a_1 m + a_2 n, b_1 m + b_2 n} M_{c_1 m + c_2 n, d_1 m + d_2 n} \eta_0 : m, n \in \mathbb{Z}\} \tag{5.1}$$

is a Riesz basis sequence in $L^2(\mathbb{R}^2)$. Then the operators $H$ with spreading functions belonging to the closed linear span of the above family (5.1) have the following series representation

$$H = \sum_{m,n} c_{m,n} H_{m,n}$$

in terms of the Riesz basis of operators

$$H_{m,n} = T_{(a_1-d_1)m+(a_2-d_2)n} M_{c_1 m + c_2 n} H_0 T_{d_1 m + d_2 n} M_{(b_1-c_1)m+(b_2-c_2)n}.$$

We noted already that if $\mathcal{H} = \{H \in \overline{\operatorname{span}}\, H_{m,n}\}$ is identifiable with $f$, then $\{H_{m,n} f\}$ is a Riesz basis sequence.

The numerical examples from Sections 4.4 and 4.5 showed that the *relevant density measure* of the system, used in conjuction with Lemma 4.7, arises from the density of lattice

$$\Lambda' = \begin{pmatrix} a_1 - d_1 & a_2 - d_2 \\ c_1 & c_2 \end{pmatrix} \mathbb{Z}^2. \tag{5.2}$$

The lattice $\Lambda'$ parametrizes the time-frequency shifts $T_{(a_1-d_1)m+(a_2-d_2)n} M_{c_1 m + c_2 n}$ which appear in $H_{m,n}$.

*Note*: In fact, the quantities $\mathbf{d}(\Lambda)$ and $D_{(2)}(\Lambda)$ are not strictly correlated with respect to 1, i.e. it might happen that $\mathbf{d}(\Lambda) < 1, D_{(2)}(\Lambda) > 1$ and vice versa. We provide an example of this using (4.2). First, set $a_1 = d_1 = 0.5, a_2 = 0, d_2 = 1, c_1 = c_2 = 1 + \epsilon$, with $\epsilon > 0$. Then $\mathbf{d}(\Lambda) = \frac{1}{1+\epsilon} < 1$, and $D_{(2)}(\Lambda) = \left(0.25 + \frac{(1+\epsilon)^2}{2}\right)^{-\frac{1}{2}}$. For $\epsilon < \sqrt{\frac{3}{2}} - 1$, $D_{(2)}(\Lambda) > 1$.

Second, set $a_1 = 10, a_2 = 9.5, d_2 = 1, d_1 = c_1 = c_2 = 0.5$. Then $\mathbf{d}(\Lambda) = 2 > 1$, but $D_{(2)}(\Lambda) = (5.25^2 + 0.5 \times 0.25)^{-\frac{1}{2}} < 1$.

The relevant 2-density for the study of the operator sequence $\{H_{m,n}\}$ is therefore $\mathbf{d}(\Lambda) = |\Lambda'|^{-1}$, which does not involve the coefficients $b_1, b_2$ from the formula for the 2-density of $\Lambda$. That is why, without loss of generality, we can assume that $b_1 = b_2 = 0$ for the remainder of our discussion. Then the 2-density of the original system of points according to (4.2) is given by

$$D_{(2)}(\Lambda) = ((a_1c_2 - a_2c_1)^2 + (a_1d_2 - a_2d_1)^2 + (c_1d_2 - c_2d_1)^2)^{-\frac{1}{2}}. \tag{5.3}$$

In order to analyze operator families with spreading functions with expansions in terms of the family like (5.1) we use the toolbox of Gabor molecules and localization of function sequences [BCHL06a], [BCHL06b].

We recall the most important definitions from this theory. The first one will be the notion of a density of a point set with respect to a map.

Let $\mathcal{I}$ be a countable index set, and $G = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, and $a : \mathcal{I} \to G$ a map. For every $n \in \mathbb{N}$ we denote the box with size $n$ centered at $j \in G$ by $S_n(j) := \{g \in G : \|g-j\|_\infty \le \frac{n}{2}\}$. The cardinality of $S_n(j)$ is independent of $j$ since $G$ is a group. In general $\lim\limits_{n\to\infty} \frac{|S_n(j)|}{n^d} = \frac{1}{(\alpha\beta)^d}$ (which is the Beurling density of $G$). Let $\mathcal{I}_n(j)$ be the pre-image of $S_n(j)$ under $a$, in other words $\mathcal{I}_n(j) = a^{-1}(S_n(j))$.

**Definition 5.1** *The lower and upper densities of $\mathcal{I}$ with respect to $a$ are*

$$\begin{aligned} D_a^-(\mathcal{I}) &= \liminf_{n\to\infty} \inf_{j\in G} \frac{|\mathcal{I}_n(j)|}{|S_n(j)|}, \\ D_a^+(\mathcal{I}) &= \limsup_{n\to\infty} \sup_{j\in G} \frac{|\mathcal{I}_n(j)|}{|S_n(j)|}. \end{aligned} \tag{5.4}$$

These quantities can be 0 or infinite. When $D_a^-(\mathcal{I}) = D_a^+(\mathcal{I})$, $\mathcal{I}$ is said to have uniform density.

When $\mathcal{I}$ is a lattice $\Lambda \subset \mathbb{R}^{2d}$ we set $a : \Lambda \to G$ to be the rounding function:

$$a(x, \omega) = (\lfloor \frac{x}{\alpha} \rfloor, \lfloor \frac{y}{\beta} \rfloor), \quad (x, \omega) \in \Lambda.$$

In this case there is a relation between the $a$-density and the Beurling density (4.1) of $\Lambda$, namely

$$\begin{aligned} D^-(\Lambda) &= \frac{|D_a^-(\Lambda)|}{(\alpha\beta)^d} \\ D^+(\Lambda) &= \frac{|D_a^+(\Lambda)|}{(\alpha\beta)^d} \end{aligned} \tag{5.5}$$

For the computation of the above formulas we refer to [BCHL06a], p. 113.

Localization can be defined in terms of decay of the inner products of one sequence $\mathcal{G}$ with members of another sequence $\mathcal{E}$. In fact, these inner products $\{\langle g, e_a\rangle : g \in \mathcal{G}, e \in \mathcal{E}\}$ are entries of a cross-Grammian matrix associated to the triple $(\mathcal{G}, a, \mathcal{E})$. Its rows and columns can be required to possess a certain decay. Unlike [BCHL06a], we will be interested only in row decay. Gabor molecules are a particular example of this definition as we shall see in Definition 5.5.

**Definition 5.2 (Localization, [BCHL06a], Def. 3)** *Let* $\mathcal{G} = \{f_i : i \in \mathcal{I}\}, \mathcal{E} = \{e_j : j \in G\}$ *be sequences in a Hilbert space* $\mathcal{H}$*, and* $a : \mathcal{I} \to G$ *an associated map.* $(\mathcal{G}, a, \mathcal{E})$ *is* $\ell^p$*-localized* ($1 \leq p < \infty$) *if*

$$\sum_{j\in G} \sup_{i\in\mathcal{I}} |\langle f_i, e_{j+a(i)}\rangle|^p < \infty.$$

*Equivalently, there must exist* $\mathbf{c} \in \ell^p(G)$ *such that for all* $i \in \mathcal{I}, j \in G$,

$$|\langle f_i, e_j\rangle| \leq c_{a(i)-j}.$$

*Furthermore, the cross-Grammian matrix of* $(\mathcal{G}, a, \mathcal{E})$ *has* $\ell^p$*-row decay if for every* $\varepsilon > 0$ *there exists a non-negative integer* $N_\varepsilon$ *such that for all* $i \in \mathcal{I}$

$$\sum_{j\in G\setminus S_{N_\varepsilon}(a(i))} |\langle f_i, e_j\rangle|^p < \varepsilon.$$

**Note:** The localization of $\mathcal{G}$ as defined by Definition 5.2 depends on the choice of $\mathcal{E}$ and $a$. In fact $\ell^p$-localization implies $\ell^p$-row decay of the the cross-Grammian matrix associated to the triple $(\mathcal{G}, a, \mathcal{E})$, as shown by the following simple computation.

Let $\varepsilon > 0$ be given. Assume $(\mathcal{G}, a, \mathcal{E})$ is $\ell^p$-localized. Fix $i \in \mathcal{I}$. Choose $N_\varepsilon$ such that

$$\sum_{k\in G\setminus S_{N_\varepsilon}(0)} c_k^p < \varepsilon.$$

Then a simple change of variables in the sum produces

$$\begin{aligned}\sum_{j\in G\setminus S_{N_\varepsilon}(a(i))} |\langle f_i, e_j\rangle|^p &\leq \sum_{j\in G\setminus S_{N_\varepsilon}(a(i))} c_{a(i)-j}^p \\ &\leq \sum_{k\in G\setminus S_{N_\varepsilon}(0)} c_k^p < \varepsilon\end{aligned}$$

Next comes a generalized version of Homogeneous Approximation Properties [CBH99], [RS95], which are characteristic of Gabor frames. The generalized HAP does not involve the structure of Gabor frames.

**Definition 5.3 (Dual HAP [BCHL06a], Def. 4)** *Let* $\mathcal{G} = \{f_i : i \in \mathcal{I}\}, \mathcal{E} = \{e_j : j \in G\}$ *be sequences in a Hilbert space* $\mathcal{H}$ *such that* $\mathcal{E}$ *is a frame for* $\mathcal{H}$ *with dual frame* $\tilde{\mathcal{E}} = \{\tilde{e}_j : j \in G\}$ *and* $a : \mathcal{I} \to G$ *an associated map.*

1. $(\mathcal{G}, a, \mathcal{E})$ *has the weak dual HAP if for every* $\varepsilon > 0$ *there exists a non-negative integer* $N_\varepsilon$ *such that for all* $i \in \mathcal{I}$ $\varepsilon > 0$,

$$\operatorname{dist}(f_i, \overline{\operatorname{span}}\,\{\tilde{e}_j : j \in S_{N_\varepsilon}(a(i))\}) < \varepsilon.$$

2. $(\mathcal{G}, a, \mathcal{E})$ *has the strong dual HAP if for every* $\varepsilon > 0$ *there exists a non-negative integer* $N_\varepsilon$ *such that for all* $i \in \mathcal{I}$ $\varepsilon > 0$,

$$\|f_i - \sum_{j \in S_{N_\varepsilon}(a(i))} \langle f_i, e_j \rangle \tilde{e}_j\|_{\mathcal{H}} < \varepsilon.$$

Note: If $\mathcal{E}$ is a frame for $\mathcal{H}$, then the strong dual HAP implies the weak HAP ([BCHL06b], Theorem 10). If the reference system $\mathcal{E}$ is a frame for $\mathcal{H}$, then $\ell^p$-row decay implies the strong dual HAP ([BCHL06b], Theorem 10).

These Harmonic Approximation Properties will allow us to put bounds on frame densities as stated in Theorem 3, [BCHL06a].

**Theorem 5.4 ([BCHL06a])** *Assume* $\mathcal{G} = \{f_i : i \in \mathcal{I}\}$ *is a Riesz basis sequence in* $\mathcal{H}$, *and* $\mathcal{E} = \{e_j : j \in G\}$ *is a frame for* $\mathcal{H}$. *If* $(\mathcal{G}, a, \mathcal{E})$ *has the weak dual HAP, then*

$$0 \leq D_a^-(\mathcal{I}) \leq D_a^+(\mathcal{I}) \leq 1.$$

These results recapture the fact that whenever the reference system $\mathcal{E}$ is a frame for $\mathcal{H}$ and $(\mathcal{G}, a, \mathcal{E})$ is $\ell^p$-localized, then we can apply Theorem 5.4 to make estimates about the $a$-density of the index set of the Riesz basis sequence $\mathcal{G}$.

We have noted already that in order for the system of operators $\mathcal{H}_\Lambda$, where

$$\Lambda = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}^T \mathbb{Z}^2 \tag{5.6}$$

to be identifiable with $f$, the system of functions $\mathcal{G} = \{H_{m,n}f : m, n \in \mathbb{Z}\}$, with

$$H_{m,n}f = T_{(a_1-d_1)m+(a_2-d_2)n} M_{c_1 m + c_2 n} H_0 T_{d_1 m + d_2 n} M_{(b_1-c_1)m+(b_2-c_2)n} f, \tag{5.7}$$

must constitute a Riesz basis sequence in $L^2(\mathbb{R})$ as required by condition (IV). Then a result such as Theorem 5.4 will allow us to put bounds on the quantity $\mathbf{d}(\Lambda) = |\Lambda'|^{-1}$ (5.2), which is the index set of this $\mathcal{G}$.

We are free to choose a reference system $\mathcal{E} = (\gamma_1, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ generated by the Gaussian $\gamma_1$. This system is a Gabor frame for $L^2(\mathbb{R})$ for any $\alpha, \beta > 0$ with $\alpha\beta < 1$ [Lyu92], [SW92]. It remains to show that $(\mathcal{G}, a, \mathcal{E})$ is $\ell^p$-localized for some $p$. For that we will show that $\mathcal{G}$ is a set of Gabor molecules.

**Definition 5.5 (Gabor molecules)** *Let* $\mathcal{J} \subset \mathbb{R}^2$ *and* $f_j \in L^2(\mathbb{R}), j \in \mathcal{J}$ *be given. Then* $\{f_j : j \in \mathcal{J}\}$ *is a set of Gabor molecules if there exists an envelope function* $\Gamma \in W(C, \ell^2)$ *(Definition 2.4) such that for all* $j \in \mathcal{J}, z \in \mathbb{R}^2, |V_{\gamma_1} f_j(z)| \leq \Gamma(z - j)$.

Properties of Gabor molecules are presented and discussed in [BCHL06a]. If the set of functions is actually of the form $\pi(j)f_j$, the following equivalent definition may be used: $\{\pi(j)f_j : j \in \mathcal{J}\}$ is a set of Gabor molecules if there exists an envelope function $\Gamma \in W(C, \ell^2)$ such that for all $j \in \mathcal{J}, z \in \mathbb{R}^{2d}, |V_{\gamma_1} f_j(z)| \leq \Gamma(z)$.

The restriction $\eta_H \in M^1_m(\mathbb{R}^{2d})$ assures that the action of the operator $H$ onto distributions from $M^\infty(\mathbb{R}^d)$ acts in a sense as a localizer. Among other things, the study [BCHL06b], [BCHL06a] lists criteria on determining the density of $\mathcal{J}$ if the set of Gabor molecules $\{f_j : j \in \mathcal{J}\}$ is a frame, orthonormal basis or a Riesz basis for $L^2(\mathbb{R}^d)$. To apply this resault we have to demonstrate that the set $\mathcal{G}$ in (5.7) is actually a set of Gabor molecules. This is the purpose of

**Lemma 5.6** *Let $H_0$ be a prototype operator with spreading function $\eta_0 \in M^1_v(\mathbb{R}^2)$, $v$ a polynomial weight of degree strictly greater than 2, $f \in M^\infty(\mathbb{R})$. Then*

$$\mathcal{G} = \{T_{(a_1-d_1)m+(a_2-d_2)n} M_{c_1m+c_2n} H_0 T_{d_1m+d_2n} M_{(b_1-c_1)m+(b_2-c_2)n} f : m, n \in \mathbb{Z}\}$$

*is a set of Gabor molecules.*

*Proof.* We shall show that for $f \in M^\infty(\mathbb{R})$, the set $\mathcal{G}$ as given by (5.7) (whose elements we denote for short $T_{(a_1-d_1)m+(a_2-d_2)n} M_{c_1m+c_2n} f_{m,n}$) consists of Gabor molecules. Under the given assumptions Lemma (4.4) shows that

$$\begin{aligned} |f_{m,n}(x)| &= |H_0 T_{d_1m+d_2n} M_{(b_1-c_1)m+(b_2-c_2)n} f(x)| = O(|x|^{-s}), \\ |\mathcal{F} f_{m,n}(\xi)| &= |\mathcal{F} H_0 T_{d_1m+d_2n} M_{(b_1-c_1)m+(b_2-c_2)n} f(\xi)| = O(|\xi|^{-s}) \end{aligned} \tag{5.8}$$

where $s > 2$. To prove our claim we must show that there exists $\Gamma \in W(C, \ell^2)$, such that $|V_{\gamma_1} f_{m,n}(\mathbf{z})| < \Gamma(\mathbf{z})$ for all $m, n \in \mathbb{Z}$ and all $\mathbf{z} \in \mathbb{R}^2$ - see Definition 10 from [BCHL06b]. Following the reasoning and computations from have shown in Proposition 4.13, in particular (4.40), we see that

$$\begin{aligned} |V_{\gamma_1} f_{m,n}(\mathbf{z})| &= |\langle f_{m,n}, T_x M_\omega \gamma_1 \rangle| \\ &\leq \phi_1 * \gamma_1(x) \|f\|_{M^\infty} \\ |V_{\gamma_1} f_{m,n}(\mathbf{z})| &= |\langle \mathcal{F} f_{m,n}, M_x T_\omega \gamma_1 \rangle| \\ &\leq \phi_2 * \gamma_1(\omega) \|f\|_{M^\infty} \end{aligned} \tag{5.9}$$

where $\phi_1(x) = O(|x|^{-s}), \phi_2(\omega) = O(|\omega|^{-s}), s > 2$. Hence, if we set

$$h(y) = \|f\|_{M^\infty} \max\{\phi_1 * \gamma_1(y), \phi_1 * \gamma_1(-y), \phi_2 * \gamma(y), \phi_2 * \gamma(-y)\},$$

we obtain that

$$V_{\gamma_1} f_{m,n}(\mathbf{z}) \leq h(\max\{|x|, |\omega|\}) = h(\|\mathbf{z}\|_\infty)$$

and $|h(\mathbf{z})| = O(|\mathbf{z}|^{-s}), s > 2$. This means that there exists a constant $c$ such that

$$|V_{\gamma_1} f_{m,n}(\mathbf{z})| \leq c \cdot h(|\mathbf{z}|)$$

Then we can bound this function $h$ by some $\Gamma \in W(C, \ell^2)$ (see Definition 2.4) and obtain the necessary decay. Thus, the set $\mathcal{G}$ consists of Gabor molecules. □

**Lemma 5.7** *Let $H_0$ be a prototype operator with spreading function $\eta_0 \in M^1_v(\mathbb{R}^2)$, $v$ a polynomial weight of degree greater than 2, $f \in M^\infty(\mathbb{R})$. If the system*

$$\mathcal{G} = \{T_{(a_1-d_1)m+(a_2-d_2)n} M_{c_1m+c_2n} H_0 T_{d_1m+d_2n} M_{(b_1-c_1)m+(b_2-c_2)n} f : m, n \in \mathbb{Z}\}$$

*which is associated to the sampling set $\Lambda$ is a Riesz basis sequence, then $\mathbf{d}(\Lambda) \leq 1$.*

*Proof.* Lemma 5.6 shows that $\mathcal{G}$ is a set of Gabor molecules. Assume that $\mathcal{G}$ is a Riesz basis sequence. We choose $\alpha, \beta > 0$ such that $\alpha\beta < 1$ and employ as a reference system $\mathcal{E} = (\gamma_1, \alpha\mathbb{Z} \times \beta\mathbb{Z})$, which is a Gabor frame for $L^2(\mathbb{R})$. Theorem 8a from [BCHL06b] shows that $(\mathcal{G}, a, (\gamma_1, \alpha\mathbb{Z} \times \beta\mathbb{Z}))$ is $\ell^2$-localised. Hence, $(\mathcal{G}, a, (\gamma_1, \alpha\mathbb{Z}\times\beta\mathbb{Z}))$ has the weak dual HAP. Theorem 5.4 and (5.5) in combination allow us to conclude that

$$1 \geq D^+_a(\Lambda) = (\alpha\beta) D^+_B(\Lambda). \tag{5.10}$$

However, the upper Beurling density of the index set of $\mathcal{G}$, $D^+_B(\Lambda)$, equals simply $\mathbf{d}(\Lambda)$. Since (5.10) holds for any $0 < \alpha\beta < 1$, then necessarily $\mathbf{d}(\Lambda) \leq 1$. □
Lemma 5.7 is used to demonstrate the following:

**Theorem 5.8** *If the system of operators $\mathcal{H}_\Lambda$ arising from an index set $\Lambda$ (5.6) and prototype spreading function from $M^1_v(\mathbb{R}^2)$, $v$ a polynomial weight of degree greater than 2 is identifiable, then the 2-density of $\Lambda$ must be less than $\sqrt{2}$.*

*Proof.* Lemma 5.7 shows that under the given assumptions, $\mathcal{H}_\Lambda$ is identifiable implies that $d(\Lambda) \leq 1$. This means in terms of the formula for $d(\Lambda)$ that $|(a_1 - d_1)c_2 - (a_2 - d_2)c_1| \geq 1$. In other words, $|(a_1c_2 - a_2c_1) + (c_1d_2 - c_2d_1)| \geq 1$. Then by Cauchy-Schwarz inequality:

$$(a_1c_2 - a_2c_1)^2 + (c_1d_2 - c_2d_1)^2 \geq \frac{1}{2}|(a_1c_2 - a_2c_1) + (c_1d_2 - c_2d_1)|^2 \geq \frac{1}{2}.$$

This implies by the formula for 2-density that $D_2(\Lambda) \leq \sqrt{2}$. The bound is attained for coefficients satisfying for instance $c_1a_2 = c_2a_1, a_2c_1 - c_2a_1 = d_1c_2 - d_2c_1$. □
In other words, we have demonstrated the existence of a lower bound $C$ in

**Theorem 5.9** *Let $\mathcal{H}_\Lambda = \{H : \eta_H \in \overline{\text{span}}\,(\eta_0, \Lambda)\}$, where $\eta_0 \in M^1_v(\mathbb{R}^2)$ is a prototype spreading function, $v$ a polynomial weight of degree greater than 2, and $\Lambda$ a 2-dimensional time-frequency index set inside $\mathbb{R}^4$. If $D_2(\Lambda) > \sqrt{2}$, then the operator family $\mathcal{H}_\Lambda$ is not identifiable with $f \in M^\infty(\mathbb{R})$.*

*Remark*: This value of $C$ is a universal constant. For some $\Lambda$, $C$ may have a lower value.

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